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PROPERTIES OF BOOLEAN FUNCTIONS AND HAMMING DISTANCE

by

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ABSTRACT

Previously, the study of the methods to predict the properties of Boolean functions has concentrated on the Boolean function itself; the properties of a Boolean function have been derived by manipulating and analysing this Boolean function. In this thesis, the use of Hamming distance to predict various properties of Boolean functions, is investigated. Several parameters, such as the total mutual distance among the vertices of a function and the largest distance between two vertices in a function, are defined. It is found that such parameters characterize the properties of a Boolean function $F(x_1,\ldots,x_n)$. For example, if the number of vertices in F, M, is not greater than 2^{n-1} and D, the largest Hamming distance in F, is less than n, then F is dual-comparable. If M is not greater than 2^{n-1} and D equals n, then F is not completely monotonic. Five tables, which list the calculated mutual distance totals for Boolean functions of three to seven variables, are included; these tables can be used to help determine the monotonicity and dual-comparability of F.

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TABLE OF CONTENTS

Table of Contents, continued

LIST OF TABLES

LIST OF FIGURES

CHAPTER I

INTRODUCTION

It is increasingly important that electronic computers become capable of greater speed. Consequently, logical devices, that can perform complex operations themselves, are desirable. Perhaps, devices based on the threshold principle, such as parametrons (1) and Esaki diode circuits (2) may be the answer. The Boolean function realized by a single threshold device is relatively complex compared to the Boolean function that can be realized by either an "AND," an "OR," or a "NOT" gate; generally, the number of threshold logic elements required to realize a Boolean function is considerably less than the number of ordinary gates needed to realize this same Boolean function (3). Therefore, the proper use of threshold devices in computers may lead to valuable increases in speed and savings in equipment.

Threshold elements are of interest in the memory-

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(1) See Gschwind (1967: 469-474).

(2) See Gschwind (1967: 507-514).

(3) See Winder (1961) and Ghosh (1969).

less processing of information. The mathematical models of threshold gates can be applied to such disciplines as decision theory and adaptive control; also, approximate mechanizations of these threshold models are found in the nervous systems of animals. Hence, the interest in threshold logic elements is not restricted to logical designers.

In this thesis, the use of Hamming distance to predict various properties of Boolean functions is investigated. Initially, Hamming distance between two vertices is defined. In a Boolean n-space, there are $2^{n-1}(2^n-1)$ distinct combinations of two vertices; hence, there are $2^{n-1}(2^n-1)$ possible Hamming distances in a Boolean n-space. For a Boolean function $F(x_1, \ldots, x_n)$ and two vertices, which are n-tuples, there are three possibilities; both vertices lie in F, both vertices lie in $\overline{F}$, or one vertex lies in F and one vertex lies in $\overline{F}$. Three distance vectors $d = (d_1, \ldots, d_n)$, $\overline{d} = (\overline{d}_1, \ldots, \overline{d}_n)$, and $d' = (d'_1, \ldots, d'_n)$ are defined, such that, for each vector, the i^{th} element, $1 \leq i \leq n$, represents the number of Hamming distances of i. Vector d records all distances of distinct combinations of two vertices lying in F. Vector $\overline{d}$ records all distances of distinct combinations of two vertices lying in $\overline{F}$. Vector d' records all distances of distinct combinations of two vertices where one vertex lies in F and one vertex lies in $\overline{F}$. Another parameter called the

mutual distance total Q of a Boolean function $F(x_1, \ldots, x_n)$, which is the sum of all the possible Hamming distances of F, may be easily calculated from the distance vector d. Hence, three distance vectors and a mutual distance total Q may be calculated for a Boolean function $F(x_1, \ldots, x_n)$, These distance parameters describe, to a significant extent, the structure of F. The objective of this thesis is to discover appropriate theorems on how various properties of a Boolean function $F(x_1, \ldots, x_n)$ are necessarily implied from these distance parameters.

Before going into any detail about the properties of a Boolean function, it is necessary to clarify some basic definitions and notation which will be used in this thesis.

A binary variable x_i is a variable which assumes the value of either element of the two element set B, i.e., $x_i \in B = \{0, 1\}$.

A literal is a variable not specified as to whether it is in the unnegated form, x_i, or in the negated form, $\bar{x}_i$; it is denoted by x_i.

A simple product is a conjunction or product of a subset of the literals $x_1, \ldots, x_n$; for instance, $x_2\bar{x}_4x_5$ is a simple product.

A minterm, fundamental product, or n-tuple is a

simple product of all the n literals; for instance, $x_1 \ \bar{x}_2 \cdots x_{n-1} \ \bar{x}_n$ is a minterm.

A switching or Boolean function is a function of n binary variables such that it assumes a value of either 0 or 1 when each of its n variables assumes a value of 0 or 1. Notice that, given a Boolean function $F(x_1, \ldots, x_n) = 0$ and a Boolean function $G(x_1, \ldots, x_n) = 1$, F and G are functions of n variables; they are independent of all the n variables and are called constant functions.

A Boolean expression, form, or representation, is a particular form in which a Boolean function is expressed. It is built up from the variables and the constants 0 and 1 by a finite number of applications of the Boolean operations of addition (+), multiplication (·), and complementation ($\bar{\ }$).

A normal disjunctive form or a sum-of-products form is a disjunction or sum of simple products.

A disjunct is a simple product appearing in a normal disjunctive form.

A disjunctive canonical form of a Boolean function is a normal disjunctive form whose disjuncts are all distinct minterms.

An implicant P of a function F is a simple product that can be found in some normal disjunctive form of F.

An implicant P of a function F is called a prime implicant if P implies F, and no other implicant P' exists such that P implies P'.

A prime implicant P of a function F is called an essential prime implicant if P contains at least one vertex which is not contained in any other prime implicant of F.

An irredundant normal disjunctive form is a representation of a Boolean function as a disjunction of a subset of its prime implicants such that no prime implicant is redundant or may be deleted from the expression.

A minimum sum-of-products form is a representation of a Boolean function as a disjunction of a subset of its prime implicants such that it is minimum according to a criterion, say the number of terms in the expression and then the number of literals. Note that a minimum sum-of-product form is necessarily an irredundant normal disjunctive form, but an irredundant normal disjunctive form is not necessarily a minimum sum-of-product form.

The Cartesian product of n copies of B, $B^n = BxBx\cdots xB$, is called the n-cube. There are 2^n elements or vertices in the n-cube, they are the 2^n different

valuations on the ordered n-tuple $X = (x_1,\ldots,x_n)$.

In an n-cube, the subset of vertices corresponding to a Boolean product of (n-r) literals, $0 \leq r \leq n$, where each literal is present as a factor exactly once, either complemented or not, is called an r-subcube. A don't care symbol * is used to replace variable x_j where x_j is not present in the product. For instance, in a 6-cube, the 3-cube corresponding to $x_1 x_3 \overline{x}_4$ is represented as 1*10**. There are 2^r vertices in an r-cube.

A Boolean function $F(x_1,\ldots,x_n)$ is a threshold function if there exists a vector $A = (a_1,\ldots,a_n)$ of real numbers and a threshold value T such that $XA^t = T$ represents a hyperplane, in the n-dimensional Euclidean space, which separates F and $\overline{F}$; the realization of F is $R[F] = (a_1,\ldots, a_n;T)$.

The notation used in this thesis, in some cases, is mixed notation for Boolean and set-theoretic operations and relations. However, most of them are just notations used in Boolean functions or ordinary algebra. e.g. + denotes Boolean addition and ordinary algebraic addition.

- $\cdot$ denotes Boolean multiplication and ordinary algebraic multiplication.
- $-$ denotes complementation.
- $=$ denotes equivalence.
- $\subset$ denotes the upper bound property for sets and switching functions. It reads "lies in" and is used in place of "implies."

$\supset$ denotes the lower bound property for sets and switching functions. It reads "contains" and is used in place of "is implied by."

$F_1 \subseteq F_2$ denotes that the two Boolean functions F_1 and F_2 satisfy the relation $F_1 \subset F_2$ or the relation $F_1 = F_2$.

$F_1 \supseteq F_2$ denotes that the two Boolean functions F_1 and F_2 satisfy the relation $F_1 \supset F_2$ or the relation $F_1 = F_2$.

$\leq$ denotes algebraic upper bound property.

$\geq$ denotes algebraic lower bound property.

iff denotes "if and only if."

ϵ denotes "is a member of."

$\notin$ denotes "is not a member of."

M(F) denotes the number of vertices in a Boolean function F.

$F_1 : F_2$ denotes that the two Boolean functions F_1 and F_2 are comparable; that is, $F_1 \subseteq F_2$ or $F_1 \supset F_2$.

CHAPTER 2

BOOLEAN FUNCTIONS

2.1 Characterization of Boolean Functions

2.1.1 Reduced Functions

A Boolean function $F(x_1,...,x_n)$ can be expanded along any variable x_i, $1 \le i \le n$, such that $F = x_i F_{x_i} + \bar{x}_i F_{\bar{x}_i}$. The reduced function F_{x_i} is defined to be $F(x_1,...,x_{i-1},1,x_{i+1},...,x_n)$; the reduced function $F_{\bar{x}_i}$ is defined to be $F(x_1,...,x_{i-1}, 0, x_{i+1},...,x_n)$. It is possible to further expand F along another variable x_j, $1 \le i < j \le n$, such that

$$F = x_i x_j F_{x_i x_j} + \bar{x}_i x_j F_{\bar{x}_i x_j} + x_i \bar{x}_j F_{x_i \bar{x}_j} + \bar{x}_i \bar{x}_j F_{\bar{x}_i \bar{x}_j} .$$

In fact, F can be expanded along all of its n variables.

2.1.2 Chow Parameters

Chow proved that a set of (n+1) parameters can uniquely characterize a threshold function; consequently, these parameters are called Chow parameters (4). Let a

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(4) See Chow (1961b).

Boolean function $F(x_1, \ldots, x_n)$ contain M vertices

$$X_j = (x_{j1}, x_{j2}, \ldots, x_{jn}),\ j=1, \ldots, M.$$

Then

$$C = (C_1, C_2, \ldots, C_n) = \sum_{j=1}^{M} X_j$$

$$\text{or} \quad C_i = \sum_{j=1}^{M} x_{ji}$$

Hence, c_i is equal to the number of vertices of F in the (n-1)-cube x_i; that is, c_i is equal to the number of vertices in $x_i F_{x_i}$. The (n+1) Chow parameters are $(M, C_1, \ldots, C_n)$; the Chow parameters are sometimes labelled $(M(F), C_1(F), \ldots, C_n(F))$.

2.2 Properties of Boolean Functions

2.2.1 Unateness

A Boolean function $F(x_1, \ldots, x_n)$ is said to be positive in a variable x_i, $1 \leq i \leq n$, iff F has a normal form in which no term has $\overline{x}_i$ as a factor. Also, F is said to be negative x_i iff F has a normal form in which no term has x_i as a factor. For instance, $F = x_1\overline{x}_3x_4 + x_2\,\overline{x}_4$ is positive in x_1 and x_2, negative in x_3, and neither positive or negative in x_4. A Boolean function F, which is positive in x_i and containing a minterm $x_1 \cdots x_{i-1}\overline{x}_i x_{i+1} \cdots x_n$ must also contain the minterm $x_1 \cdots x_{i-1} x_i x_{i+1} \cdots x_n$; these two minterms can form the simple product $x_1 \cdots x_{i-1} x_{i+1} \cdots x_n$ which does

not contain $\overline{x}_i$ as a factor. Geometrically, if F contains one vertex in the (n-1) cube $\overline{x}_i$, then F must also contain the corresponding vertex in the (n-1) cube x_i which differs from the vertex in the (n-1) cube $\overline{x}_i$ only in the i^{th} coordinate. These two vertices are consecutive and form a 1-cube.

Expand F along the variable x_i, $1 \leq i \leq n$, so that:

$$F = x_i F_{x_i} + \overline{x}_i F_{\overline{x}_i} .$$

If F is positive in x_i, then $F_{x_i} \supset F_{\overline{x}_i}$. If F is negative in x_i, then $F_{x_i} \subset F_{\overline{x}_i}$. If F is both positive and negative in x_i then:

$$F_{x_i} \supset F_{\overline{x}_i} \text{ and } F_{x_i} \subset F_{\overline{x}_i} ,$$

or $$F_{x_i} = F_{\overline{x}_i} .$$

Then,

$$F = x_i F_{x_i} + \overline{x}_i F_{\overline{x}_i}$$

$$= F_{\overline{x}_i} .$$

That is, if $F_{x_i} = F_{\overline{x}_i}$, then F is independent of x_i.

Proposition 2.2.1.1 (5) If a Boolean function $F(x_1,\ldots,x_n)$ is positive in x_i, it can be expressed as

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(5) See Sheng (1969: 27).

$x_i G_1 + G_2$ where G_1 and G_2 are functions independent of x_i such that $G_1 \supset G_2$.

Corollary 2.2.1.1-1. If a Boolean function $F(x_1, \ldots, x_n)$ is positive in x_i and not negative in x_i, then $C_i > M/2$.

Corollary 2.2.1.1-2. If a Boolean function $F(x_1, \ldots, x_n)$ is independent of x_i, then $C_i = M/2$.

A Boolean function $F(x_1, \ldots, x_n)$ is said to be unate in x_i iff F is either positive in x_i, negative in x_i, or independent of x_i. Also, a Boolean function $F(x_1, \ldots, x_n)$ is said to be unate iff it is unate in each of its variables (6). For instance, $F = \bar{x}_1\bar{x}_2 + \bar{x}_1 x_3 x_4 + \bar{x}_2 x_3 x_4$ is a unate function because it is negative in x_1 and x_2, and positive in x_3 and x_4.

Two Boolean functions F_1 and F_2 are said to be comparable iff either $F_1 \supseteq F_2$ or $F_1 \subset F_2$. Hence, the reduced functions F_{x_i} and $F_{\bar{x}_i}$, of a unate function $F(x_1, \ldots, x_n)$ are comparable.

Proposition 2.2.1.2. (7) A unate function $F(x_1, \ldots, x_n)$ is positive in x_i and not negative in x_i iff $C_i > M/2$.

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(6) Unate functions have been studied extensively by Dertouzos (1965: 245), Hu (1965: 68-70), Liss (1963), Paul and McCluskey (1960), McNaughton (1961), Sheng (1969: 27) and Winder (1962: 16-21).

(7) See Sheng (1969: 28).

Proposition 2.2.1.3 (8). A unate function $F(x_1, \ldots, x_n)$ is independent of x_i iff $C_i = M/2$.

Proposition 2.2.1.4 (9). If x_i occurs in a prime implicant of a Boolean function $F(x_1, \ldots, x_n)$, then F is not independent of x_i.

Corollary 2.2.1.4-1. If x_i occurs in the prime implicant of a unate function $F(x_1, \ldots, x_n)$, then F is positive in x_i, and $\bar{x}_i$ cannot occur in any prime implicant of F.

Corollary 2.2.1.4-2. The disjunction of any subset of prime implicants of a unate function $F(x_1, \ldots, x_n)$ is unate.

Corollary 2.2.1.4-3. All the prime implicants of a unate function $F(x_1, \ldots, x_n)$, considered as r-cubes, contain at least one vertex in common. For instance, if F $(x_1, \ldots, x_n)$ is positive in $x_1, \ldots, x_n$, then $x_1 \cdots x_n$ is a common vertex of all the prime implicants.

Proposition 2.2.1.5 (10). If a prime implicant P_1 of a Boolean function $F(x_1, \ldots, x_n)$ implies the disjunction of several other prime implicants $P_2, P_3, \cdots, P_M$ of F, then F

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(8) See Sheng (1969: 28).

(9) See Sheng (1969: 28).

(10) See Sheng (1969: 29).

cannot be unate.

Proposition 2.2.1.6 (11). All prime implicants of a unate function $F(x_1, \ldots, x_n)$ are essential implicants.

Corollary 2.2.1.6-1. A unate function $F(x_1, \ldots, x_n)$ has a unique irredundant normal disjunctive form which consists of the disjunction of all of the prime implicants of F.

2.2.2 Monotonicity

Monotonicity and its related properties have been studied by Hu (1965: 58-78), Paul and McCluskey (1960), Sheng (1969: 27-39), and Winder (1962). A Boolean function $F(x_1, \ldots, x_n)$ is said to be 1-monotonic iff, for all i, the reduced functions expanded along x_i (i.e. F_{x_i} and $F_{\bar{x}_i}$) are comparable. A Boolean function $F(x_1, \ldots, x_n)$ is said to be k-monotonic iff the reduced functions of F expanded along any i variables, $1 \leq i \leq k$, are comparable. Finally, a Boolean function $F(x_1, \ldots, x_n)$ is said to be completely monotonic iff it is n-monotonic. Notice that if a Boolean function is k-monotonic, $k \geq 2$, it is also i-monotonic, $1 \leq i \leq k$.

Proposition 2.2.2.1 (12) A Boolean function $F(x_1,$

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(11) See Sheng (1969: 29).

(12) See Sheng (1969: 30).

$\ldots, x_n)$ is unate iff it is 1-monotonic.

Proposition 2.2.2.2 (13) Let $F(x_1, \ldots, x_n)$ be 2-monotonic and $F_{x_j\bar{x}_k}$ and $F_{\bar{x}_j x_k}$ be two reduced functions of F expanded along any two variables x_j and x_k. Then

$$F_{x_j\bar{x}_k} \subset F_{\bar{x}_j x_k} \text{ and } F_{x_j\bar{x}_k} \not\supset F_{\bar{x}_j x_k} \text{ iff } C_j < C_k,$$

$$F_{x_j\bar{x}_k} = F_{\bar{x}_j x_k} \text{ iff } C_j = C_k.$$

Two threshold functions $F_1(x_1, \ldots, x_n)$ and $F_2(x_1, \ldots, x_n)$ are said to be isobaric iff they can be realized with the same weight vector $A = (a_1, \ldots, a_n)$ and two different threshold values T_1 and T_2, respectively. Similarly M threshold functions $F_1(x_1, \ldots, x_n)$, $F_2(x_1, \ldots, x_n)$, $\ldots$, $F_M(x_1, \ldots, x_n)$ are said to be isobaric iff they can be realized with the same weight vector $A = (a_i, \ldots, a_n)$ and M different threshold values $T_1, T_2, \ldots, T_M$, respectively.

Proposition 2.2.2.3 (14) Two isobaric threshold functions $F_1(x_1, \ldots, x_n)$ and $F_2(x_1, \ldots, x_n)$ are comparable and $F_1 \supset F_2$ if $T_1 \leq T_2$.

Corollary 2.2.2.3-1. Let $F_1, F_2, \ldots F_M$ be M

(13) See Sheng (1969: 30).

(14) See Sheng (1969: 31).

isobaric threshold functions with threshold values T_1, T_2, ..., T_M, respectively. If $T_1 \leq T_2 \leq \cdots \leq T_M$, then $F_1 \supset F_2 \supset \cdots \supset F_M$.

Proposition 2.2.2.4 (15) The reduced functions of a threshold function $F(x_1, \ldots, x_n)$ expanded along any one subset of variables, are isobaric threshold functions.

Corollary 2.2.2.4-1. Any threshold function $F(x_1, \ldots, x_n)$ is completely monotonic.

Proposition 2.2.2.5 (16) An $[n/2]$- monotonic Boolean function $F(x_1, \ldots, x_n)$ is completely monotonic, where $[n/2]$ the greatest integer $\leq n/2$.

Corollary 2.2.2.5-1. All unate functions of three or fewer variables are threshold functions.

Corollary 2.2.2.5-2. All 2-monotonic functions of five or fewer variables are threshold functions.

Proposition 2.2.2.6 (17) If two Boolean functions $F_1(x_1, \ldots, x_n)$ and $F_2\ (x_1, \ldots, x_n)$ are comparable, then the reduced functions of F_1 and F_2 for any common (n-M)-cube $x^*_{i_1} \ldots x^*_{i_M}$ (that is, $F_{1x^*_{i_1} \ldots x^*_{i_M}}$ and $F_{2x^*_{i_1} \ldots x^*_{i_M}}$)

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(15) See Sheng (1969: 31).

(16) See Sheng (1969: 32) and Winder (1962: 11).

(17) See Sheng (1969: 33).

are comparable for M=1, ..., n.

Proposition 2.2.2.7 (18) A k-monotonic Boolean function $F(x_1, \ldots, x_n)$ is (k+1)-monotonic iff for every pair of opposite (n-k-1)-cubes $x^*_{i_1} \cdots x^*_{i_{k+1}}$ and $\bar{x}^*_{i_1} \cdots \bar{x}^*_{i_{k+1}}$, the corresponding reduced functions $F_{x^*_{i_1} \cdots x^*_{i_{k+1}}}$ and $F_{\bar{x}^*_{i_1} \cdots \bar{x}^*_{i_{k+1}}}$ are comparable.

Proposition 2.2.2.8 (19) For fixed n, the following sequence of properties defines a strictly decreasing sequence of class of functions: 1-monotonicity, 2-monotonicity, ..., $[n/2]$-monotonicity = complete monotonicity.

2.2.3 Asummability

For all integral values of $k > 2$, k-assumability is a necessary and sufficient condition for threshold functions. K-asummability is derived from the concept of disjoint convex hulls; it has been studied extensively by Chow (1961a), Gabelman (1962), and Highleyman (1961).

A Boolean function $F(x_1, \ldots, x_n)$ is said to be k-summable, k being an integer ≥ 2, iff for some integer j, $2 \leq j \leq k$, there are j vertices $\{X_1, \ldots, X_j\}$ in F and j vertices $\{Y_1, \ldots, Y_j\}$ in $\bar{F}$, not necessarily distinct, such

(18) See Sheng (1969: 34).

(19) See Sheng (1969: 34) and Winder (1962: 12).

that

$$\sum_{i=1}^{j} X_i = \sum_{i=1}^{j} Y_i .$$

F is k-asummable if F is not k-summable.

Proposition 2.2.3.1 (20) A Boolean function $F(x_1, \ldots, x_n)$ is a threshold function iff it is asummable.

Proposition 2.2.3.1 (21) A Boolean function $F(x_1, \ldots, x_n)$ is completely monotonic iff it is 2-asummable.

2.3 Classification of Boolean Functions

2.3.1 Admissibly Equivalent Functions

An admissible transformation of B^n is a bijective (one-to-one and onto) transformation α: $B^n \rightarrow B^n$ which can be extended to a linear transformation of Euclidean space (22). The set of all admissible transformations of B^n forms a group called the hyper-octahedral group which is denoted by O_n. All admissible transformations correspond to permuting and complementing subsets of the n variables. The hyper-octahedral group O_n is of order $2^n (n!)$ since there are 2^n possible ways of complementing n variables and n! possible ways of permuting n variables. Every admissible transforma-

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(20) See Hu (1965: 77), Sheng (1969: 36) and Winder (1962: 38).

(21) See Hu (1965: 79), Sheng (1969: 37) and Winder (1962: 38).

(22) See Sheng (1969: 40).

tion $\alpha \in O_n$ induces a transformation $\alpha\colon \Phi \to \Phi$ of the set Φ of all Boolean functions; hence, $\alpha(F) = G$ where $F \in \Phi$ and $G \in \Phi$. For instance, let $F = x_1x_3 + x_2\bar{x}_3$ and α be the complementation of variable x_3; then, $\alpha(F) = x_1\bar{x}_3 + x_2x_3$.

Two Boolean functions $F(x_1, \ldots, x_n)$ and $G(x_1, \ldots, x_n)$ are said to be admissibly equivalent iff there exists an admissible transformation $\alpha \in O_n$ such that $\alpha(F) = G$. For instance, $F = x_1x_2 + x_3x_4$ and $G = x_2x_3 + x_1x_4$ are admissibly equivalent. Boolean functions are divided into disjoint equivalence classes which are sometimes called the symmetry types. For any complementation and/or permutation of variables (i.e., any admissible transformation α), the transformation $\alpha\colon \Phi \to \Phi$ preserves linear separability, k-summability, k-asummability, k-monotonicity, and complete monotonicity. Then every unate function $F(x_1, \ldots, x_n)$ is admissibly equivalent to a positive Boolean function.

Proposition 2.3.1.1 (23) If a threshold function $F(x_1, \ldots, x_n)$ has a realization $R[F] = (a_1, \ldots, a_n; T)$ the Boolean function $G(x_1, \ldots, x_n) = F(x_1, \ldots, x_{j-1}, \bar{x}_j, x_{j+1}, \ldots, x_n)$ is also a threshold function and has a realization $R[G] = (a_1', \ldots, a_n'; T')$ such that $a_i' = a_i$ for $i \neq j$, $a_i' = -a_j$ for $i = j$, and $T' = T - a_j$.

Corollary 2.3.1.1-1. If a threshold function $F(x_1, \ldots, x_n)$ has a realization $R[F] = (a_1, \ldots, a_n; T)$,

(23) See Sheng (1969: 41).

then the Boolean function G obtained from F by complementing a subset of variables $\{x_{i_1}, \ldots, x_{i_M}\}$ is also a threshold function and has a realization $R[G] = (a_1', \ldots, a_n'; T')$ such that $a_i' = a_i$ for $x_i \notin \{x_{i_1}, \ldots, x_{i_M}\}$, $a_i' = -a_i$ for $x_i \in \{x_{i_1}, \ldots, x_{i_M}\}$, and $T' = T - \sum_{j=1}^{M} a_{i_j}$.

Proposition 2.3.1.2. (24) If a threshold function $F(x_1, \ldots, x_n)$ has a realization $R[F] = (a_1, \ldots, a_n; T)$, then the threshold function G obtained from F by an arbitrary permutation of the variables is $R[F]$ with a permutation of the weights the same as that of the variables.

A Boolean function $F(x_1, \ldots, x_n)$ is said to be a symmetric function iff F remains invariant under any permutation of the n variables. $F(x_1, \ldots, x_n)$ is symmetric in variables x_j and x_k, $1 \leq j<k \leq n$, if $F_{x_j\bar{x}_k} = F_{\bar{x}_j x_k}$. For instance, $F = x_1x_2+x_1x_3+x_2x_3$ is symmetric in x_1 and x_2. A 2-monotonic function $F(x_1, \ldots, x_n)$ is symmetric in x_j and x_k, $1 \leq j<k \leq n$, if $c_j=c_k$.

2.3.2. Canonical Functions

A Boolean function $F(x_1, \ldots, x_n)$ is said to

(24) See Sheng (1969: 42).

to be canonical iff its Chow parameters $(M, C_1, \ldots, C_n)$ satisfy the following relations: $M/2 \leq C_1 \leq C_2 \leq \ldots \leq C_n$. In fact, every Boolean function can be changed into a canonical function by complementation and permutation of variables.

Proposition 2.3.2.1. (25) Every Boolean function $F(x_1, \ldots, x_n)$ is admissibly equivalent to at least one canonical Boolean function.

Therefore, a symmetric type or equivalent class of Boolean functions of n variables contains one, and possibly more than one, canonical Boolean function. The admissibly equivalent canonical function of an arbitrary Boolean function is not necessarily unique; however, for certain restricted classes of Boolean functions, the admissibly equivalent canonical Boolean function is unique.

Theorem 2.3.2.2. (26) Every 2-monotonic Boolean function $F(x_1, \ldots, x_n)$ is admissibly equivalent to a unique canonical Boolean function.

Corollary 2.3.2.2-1. Every threshold function $F(x_1, \ldots, x_n)$ is admissibly equivalent to a unique canonical Boolean function.

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(25) See Hu (1965: 97) and Sheng (1969: 43).

(26) See Hu (1965: 97) and Sheng (1969: 43).

2.3.3 Dual-Comparable and Self-Dual Functions

Proposition 2.3.3.1. (27) If $F(x_1, \ldots, x_n)$ is a threshold function, then the complementary function $\overline{F}(x_1, \ldots, x_n)$ is also threshold.

The complementation of a Boolean function preserves linear separability, k-summability, k-asummability, k-monotonicity, and complete monotonicity. The dual Boolean function $F^D(x_1, \ldots, x_n)$ of a Boolean function $F(x_1, \ldots, x_n)$ is defined as $F^D(x_1, \ldots, x_n) = \overline{F}(\overline{x}_1, \ldots, \overline{x}_n)$. That is, F^D is the complementary function $\overline{F}$ with all the variables complemented again. Since the complementary function $\overline{F}$ can be obtained from any expression of F by interchanging addition and multiplication operations and the complementation of the variables, the dual function can be obtained from F by simply interchanging the addition and multiplication operations.

Proposition 2.3.3.2. (28) If $F(x_1, \ldots, x_n)$ is a threshold function, then $F^D(x_1, \ldots, x_n)$ is also a threshold function.

A Boolean function F is said to be dual-comparable iff $F \supseteq F^D$ or $F \subseteq F^D$; F is said to be self-dual iff $F=F^D$.

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(27) See Sheng (1969: 54).

(28) See Sheng (1969: 56).

Corollary 2.3.3.2-1. (29) Every threshold function is dual-comparable.

2.4 Linearly Separable Boolean Functions

2.4.1. Definition of a Threshold Function

A Boolean function $F(x_1, \ldots, x_n) = F(X)$ is said to be a threshold function if the following conditions are satisfied:

$$F(X) = 1, \text{ if } f_A(X) = \sum_{i=1}^{n} a_i x_i \geq T$$

and $$F(X) = 0, \text{ if } f_A(X) = \sum_{i=1}^{n} a_i x_i < T$$

where

x_i = a binary variable assuming a value of either 0 or 1, for i=1, ..., n,

X = a vertex of the n-cube,

a_i = a real coefficient called the weight of x_i, for i=1, ..., n,

$A = (a_1, \ldots, a_n)$, the weight vector,

T = a constant called the threshold value,

$F(X)$ = a Boolean function of X,

and $f_A(X)$ = an algebraic function of X. (30)

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(29) See Muroga (1961), Muroga (1965a), Sheng (1969: 58), Winder (1965a) and Winder (1965b).

(30) See Sheng (1969: 6), Hu (1965: 28-32), Dertouzos (1965: 9-10), and Winder (1962: 2-3).

A threshold function is often called a linearly separable function, a 1-realizable function, a linear-input function, or a majority input function. A threshold logic element is a physical realization of a threshold function.

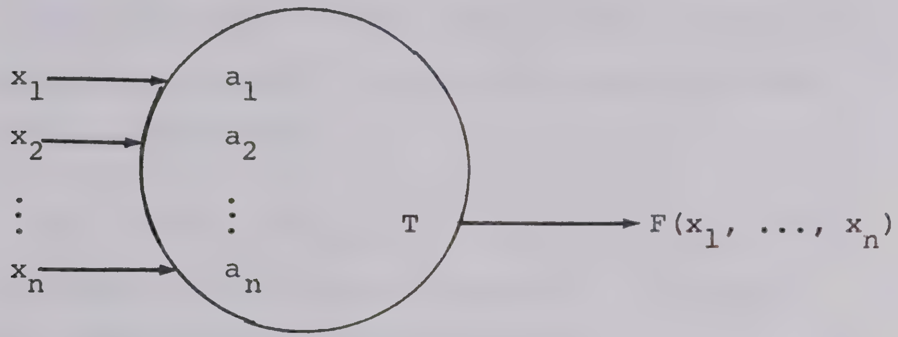

Figure 1. Symbol for a Threshold Logic Element (31)

As Figure 1 indicates, a threshold logic element consists of n binary inputs $\{x_i\}$, i=1, ..., n, n input weights $\{a_i\}$, i=1, ..., n, a threshold value T, and one output $F(x_1, \ldots, x_n)$. The realization of F is $R[F] = (a_1, \ldots, a_n; T)$. The equation

$$XA^t = T$$

represents a hyperplane, in the n-dimensional Euclidean space, called a separating hyperplane. This hyperplane separates the vertices of F from the vertices of $\overline{F}$; hence, a threshold function is sometimes called a linearly separable function.

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(31) See Sheng (1969: 7).

2.4.2 Convex Hulls

A set C is said to be convex if, whenever X_1 and X_2 are points of C, the entire line segment between X_1 and X_2 also belongs to C (32).

The convex hull of an arbitrary set of points F in the Euclidean space R^n, H(F) is the smallest convex set which contains F (33).

The geometric concept of convex sets and convex hulls is helpful in the visualization of a threshold function; consider the following theorem.

Proposition 2.4.2.1. (34) The Boolean function F is a threshold function iff

$$H(F) \cap H(\bar{F}) = \emptyset$$

where $\emptyset$ is the empty set.

2.4.3. Number of Threshold Functions

The general problem of how many threshold functions there are for n variables is still unsolved. However, lower and upper bounds, on the number of threshold

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(32) See Sheng (1969: 11).

(33) See Sheng (1969: 11).

(34) See Sheng (1969: 12).

functions of n variables, have been given by Winder (1963), Smith (1966), Yajima (1965), and Muroga (1965b). Winder (1965b) generates all the threshold functions of seven variables; Muroga (1970) generates all the threshold functions of eight variables. Table 1 includes some of the results from Muroga (1970). Notice that the ratio of the total number of threshold functions of n variables, or R(n), to the total number of Boolean functions of n variables decreases very rapidly as n increases.

TABLE 1

NUMBER OF THRESHOLD FUNCTIONS

Number of Variables, n	Number of Boolean Functions, 2^{2^n}	Number of Threshold Functions, R(n)
0	2	2
1	4	4
2	16	14
3	256	104
4	65,536	1,882
5	$\sim 4.3 \times 10^{9}$	94,572
6	$\sim 1.8 \times 10^{19}$	15,028,134
7	$\sim 3.4 \times 10^{38}$	8,378,070,864
8	$\sim 1.16 \times 10^{77}$	17,561,539,552,946

CHAPTER 3

DISTANCE VECTORS OF A BOOLEAN FUNCTION

3.1 Distance Vector d

For a Boolean function $F(x_1, \ldots, x_n)$ composed of M vertices, the number of distinct combinations of two vertices is $M(M-1)/2$. Two vertices $X_1 = x_{11}\, x_{12} \cdots x_{1n}$ and $X_2 = x_{21}\, x_{22} \cdots x_{2n}$ are said to have a Hamming distance of k iff complementing k variables x_{1i}, $1 \leq i \leq n$, of vertex X_1 produces X_2, where each element of vertex X_1 can be complemented no more than one time. Hence, if $M<2$, then no Hamming distance can be calculated for F. For each distinct combination of two vertices of F, a Hamming distance of k, $1 \leq k \leq n$, can be obtained. These distances can be recorded with a distance vector $d = (d_1, \ldots, d_n)$ where d_k, $1 \leq k \leq n$, is the number of distinct combinations of two vertices of F that have a Hamming distance of k.

3.2 Distance Vector d and Canonical Vector ℓ as Related to Monotonicity

For a Boolean function $F(x_1, \ldots, x_n)$ with Chow parameters $(M, C_1, \ldots, C_n)$, first let ℓ_i = minimum of C_i

and $(M - C_i)$, $i = 1, \ldots, n$; then, relabel the subscripts of ℓ_i so that $\ell_1 \le \ell_2 \le \cdots \le \ell_n \le M/2$. The result is a vector $\ell = (\ell_1, \ldots, \ell_n)$ which will be called the canonical vector ℓ of F. A function $G(x_1, \ldots, x_n)$ which has Chow parameters $(M, \ell_1, \ell_2, \ldots, \ell_n)$ is called a canonical function. Let this function G be admissibly equivalent to the function F, i.e., one can be obtained from the other by complementing and/or permuting some variables. Note that the canonical vector ℓ of G is its Chow parameters. Thus, F and G have the same canonical vector ℓ. In fact, all functions which are admissibly equivalent have the same canonical vector ℓ. Moreover, consider two vertices $X_1 = x_{11}x_{12}\cdots x_{1n}$ and $X_2 = x_{21}x_{22}\cdots x_{2n}$ which have a Hamming distance of k, $1 \le k \le n$. Any subset of the n variables can be complemented and/or permuted; however, vertices X_1 and X_2, after complementation and/or permutation, will retain a Hamming distance of k. Hence, admissibly equivalent functions have the same distance vector d.

For the Boolean function $G(x_1, \ldots, x_n)$, a subset of r variables, $1 \le r \le n$, is said to contribute a value k to d_r iff there are k distinct combinations of two vertices of G where only the elements of those r variables differ.

There are 2^r reduced functions that result when G is expanded, in every possible way, along a set of r variables, $1 \le r \le n$. This set of r variables is said to pre-

serve r-monotonicity iff all of these 2^r reduced functions are comparable. In the sequel, G is assumed to be a canonical function; as admissible equivalence preserves complete monotonicity, theorems proved for G are also true for G's admissibly equivalent function F.

Theorem 3.2.1. For the Boolean function $G(x_1, \ldots, x_n)$ the maximum value of d_1 is $\sum_{i=1}^{n} \ell_i$. That is, $d_1 \leq \sum_{i=1}^{n} \ell_i$.

Proof: Assume $d_1 > \sum_{i=1}^{n} \ell_i$. Then there must exist a variable x_j, $1 \leq j \leq n$, where variable x_j contributes more than ℓ_j to d_1. That is, the number of distinct combinations of two vertices $Y_1 \cdots Y_{j-1} 1 Y_{j+1} \cdots Y_n \in G$ and $Y_1 \cdots Y_{j-1} 0 Y_{j+1} \cdots Y_n \in G$, where $Y_k \in \{0,1\}$ and $1 \leq k \leq n$, must be greater than ℓ_j.

Then, $M(G_{x_j}) > \ell_j$

But, $M(G_{x_j}) = \ell_j$, a contradiction. Q.E.D.

Theorem 3.2.2. The Boolean function $G(x_1, \ldots, x_n)$ is 1-monotonic iff $\sum_{i=1}^{n} \ell_i = d_1$.

Proof: Necessity. Assume $\sum_{i=1}^{n} \ell_i > d_1$, i.e., $\sum_{i=1}^{n} \ell_i \neq d_1$ (by theorem 3.2.1.).

There must exist a variable x_j, $1 \leq j \leq n$, that contri-

butes less than ℓ_j to d_1. That is, there is a vertex $Y_1 \cdots Y_{j-1} 1 Y_{j+1} \cdots Y_n \in G$ such that vertex $Y_1 \cdots Y_{j-1} 0 Y_{j+1} \cdots Y_n \notin G$ where $Y_k \in \{0,1\}$, $1 \le k \le n$.

Then, $G_{x_j} \not\subseteq G_{\overline{x}_j}$

But $M(G_{x_j}) = \ell_j \le M - \ell_j = M(G_{\overline{x}_j})$ or $G_{x_j} \not\supset G_{\overline{x}_j}$.

Hence, G is not 1-monotonic, a contradiction.

Sufficiency. Assume G is not 1-monotonic. There must exist a vertex $Y_1 \cdots Y_{j-1} 1 Y_{j+1} \cdots Y_n \in G$ such that vertex $Y_1 \cdots Y_{j-1} 0 Y_{j+1} \cdots Y_n \notin G$, where $Y_k \in \{0,1\}$ and $1 \le k \le n$. Then, variable x_j contributes less than ℓ_j to d_1 or (using theorem 3.2.1)

$$\sum_{i=1}^{n} \ell_i > d_1, \text{ a contradiction.} \qquad \text{Q.E.D.}$$

Corollary 3.2.2-1. The Boolean function $F(x_1, \ldots, x_n)$ is 1-monotonic iff $\sum_{i=1}^{n} \ell_i = d_1$.

Theorem 3.2.3. For the Boolean function $G(x_1, \ldots, x_n)$, the maximum value of d_2 is $\sum_{i=1}^{n} \ell_i (n-i)$; that is

$$d_2 \le \sum_{i=1}^{n-1} \ell_i (n-i).$$

Proof: Assume $d_2 > \sum_{i=1}^{n-1} \ell_i (n-i)$.

Then, for some j and k, $1 \le j < k \le n$, variables x_j and x_k contribute a value $p > \ell_j$ to d_2. There are p vertices in $G_{\overline{x}_j}$

and p vertices in G_{x_j} such that p pairs of vertices can be formed where the vertices in each pair differ only in variables x_j and x_k.

Hence, $M(G_{x_j}) = p > \ell_j$

But, $M(G_{x_j}) = \ell_j$, a contradiction. Q.E.D.

Theorem 3.2.4. Let the Boolean function $G(x_1, \ldots, x_n)$ be 1-monotonic; G is 2-monotonic iff

$$\sum_{i=1}^{n-1} \ell_i \,(n-i) = d_2.$$

Proof: Necessity. Assume $\sum_{i=1}^{n-1} \ell_i \,(n-i) > d_2$,

i.e., $\sum_{i=1}^{n-1} \ell_i (n-i) \neq d_2$ (by theorem 3.2.3.).

For some j and k, $1 \leq j < k \leq n$, variables x_j and x_k contribute a value $p < \ell_j$ to d_2. That is, there are p vertices in $G_{\overline{x}_j}$ and p vertices in G_{x_j} such that p pairs of vertices can be formed where the vertices in each pair differ only in variables x_j and x_k.

Since $M(G_{x_j}) = \ell_j > p$, there are two possible cases.

case (i): There exists a vertex in $G_{x_j x_k}$ that is not in $G_{\overline{x}_j \overline{x}_k}$, or $G_{x_j x_k} \nsubseteq G_{\overline{x}_j \overline{x}_k}$.

But, $M(G_{x_j x_k}) = M(G_{x_j}) - M(G_{x_j \overline{x}_k}) \leq M(G_{\overline{x}_j \overline{x}_k}) =$

$M(G_{\overline{x}_k}) - M(G_{x_j \overline{x}_k})$

because $M(G_{x_j}) = \ell_j \leq M/2 \leq M-\ell_k = M(G_{\bar{x}_k})$.

Hence, $G_{x_j x_k} \not\supset G_{\bar{x}_j \bar{x}_k}$

Therefore, G is not 2-monotonic, a contradiction.

case(ii): There exists a vertex in $G_{x_j \bar{x}_k}$ that is not in $G_{\bar{x}_j x_k}$, or $G_{x_j \bar{x}_k} \not\subseteq G_{\bar{x}_j x_k}$.

But, $M(G_{x_j \bar{x}_k}) = M(G_{x_j}) - M(G_{x_j x_k}) \leq M(G_{\bar{x}_j x_k}) = M(G_{x_k}) - M(G_{x_j x_k})$

because $M(G_{x_j}) = \ell_j \leq \ell_k = M(G_{x_k})$.

Hence, $G_{x_j \bar{x}_k} \not\supset G_{\bar{x}_j x_k}$

Therefore, G is not 2-monotonic, a contradiction.

Sufficiency. Assume G is not 2-monotonic. Then, either of the two following cases exist.

case (i): There exists a vertex in $G_{x_j x_k}$ that is not in $G_{\bar{x}_j \bar{x}_k}$ for some j and k where $1 \leq j < k \leq n$.

case (ii). There exists a vertex in $G_{x_j \bar{x}_k}$ that is not in $G_{\bar{x}_j x_k}$.

For either of these two cases, there can, at most, exist $(\ell_j - 1)$ pairs of vertices in G where each pair differs

only in variables x_j and x_k. Ordinarily, $\ell_j = M(G_{x_j}) = M(G_{x_j x_k}) + M(G_{x_j \bar{x}_k})$ is the maximum contribution of variables x_j and x_k to d_2; however, because case (i) or case (ii) must exist, this maximum becomes $(\ell_j - 1)$.

Hence, using theorem 3.2.3., $\sum_{i=1}^{n-1} \ell_i (n-i) > d_2$, a contradiction. Q.E.D.

Corollary 3.2.4-2. The Boolean function $F(x_1, \ldots, x_n)$ is 2-monotonic iff $\sum_{i=1}^{n} \ell_i = d_1$ and $\sum_{i=1}^{n-1} \ell_i (n-i) = d_2$.

Suppose the Boolean function $G(x_1, \ldots, x_n)$ with Chow parameters $(M, \ell_1, \ldots, \ell_n)$, where $\ell_1 \leq \ell_2 \leq \cdots \leq \ell_n \leq M/2$, is 2-monotonic. Consider any combination of three variables x_{j_1}, x_{j_2}, and x_{j_3}, where $1 \leq j_1 < j_2 < j_3 \leq n$. Then $G_{x_{j_1} \bar{x}_{j_2} x_{j_3}} \subseteq G_{\bar{x}_{j_1} x_{j_2} \bar{x}_{j_3}}$, $G_{x_{j_1} x_{j_2} \bar{x}_{j_3}} \subseteq G_{\bar{x}_{j_1} \bar{x}_{j_2} x_{j_3}}$, and $G_{x_{j_1} x_{j_2} x_{j_3}} \subseteq G_{\bar{x}_{j_1} \bar{x}_{j_2} \bar{x}_{j_3}}$ because G is 2-monotonic and the Chow parameters $(M, \ell_1, \ldots, \ell_n)$ satisfy the relations $\ell_1 \leq \ell_2 \leq \cdots \leq \ell_n \leq M/2$. The variables x_{j_1}, x_{j_2}, and x_{j_3} must contribute $p = M(G_{x_{j_1} \bar{x}_{j_2} x_{j_3}}) + M(G_{x_{j_1} x_{j_2} \bar{x}_{j_3}}) + M(G_{x_{j_1} x_{j_2} x_{j_3}})$ to d_3; that is $d_3 \geq p$.

Let $S_{j_1j_2j_3} = M(G_{x_{j_1}\overline{x}_{j_2}\overline{x}_{j_3}})$ and $r_{j_1j_2j_3} =$ $M(G_{\overline{x}_{j_1}x_{j_2}x_{j_3}})$; then $\delta_{j_1j_2j_3}$ is defined to be:

$$\delta_{j_1j_2j_3} = \begin{cases} 0 \text{ iff } S_{j_1j_2j_3} \leq r_{j_1j_2j_3} \\ S_{j_1j_2j_3} - r_{j_1j_2j_3} \text{ iff } S_{j_1j_2j_3} > r_{j_1j_2j_3}. \end{cases}$$

To calculate the parameter S, sum $\delta_{j_1j_2j_3}$ for all distinct combinations of three variables.

Theorem 3.2.5. For the 2-monotonic Boolean function $G(x_1, \ldots, x_n)$, $d_3 \leq \sum_{i=1}^{n-2} \left(\ell_i \quad \frac{(n-i)(n-i-1)}{2}\right) - S.$

Proof: Assume $d_3 > \sum_{i=1}^{n-2} \left(\ell_i \quad \frac{(n-i)(n-i-1)}{2}\right) - S.$

There must be a combination of three variables x_{j_1}, x_{j_2} and x_{j_3}, $1 \leq j_1 < j_2 < j_3 \leq n$, that contribute $b > \ell_{j_1} - \delta_{j_1j_2j_3}$ to d_3.

Then, $M(G_{x_{j_1}}) = b + \delta_{j_1j_2j_3} > \ell_{j_1}$.

But, $M(G_{x_{j_1}}) = \ell_{j_1}$, a contradiction. Q.E.D.

Theorem 3.2.6. The 2-monotonic Boolean function $G(x_1, \ldots, x_n)$ is 3-monotonic iff

$$d_3 = \sum_{i=1}^{n-2} \left(\ell_i \frac{(n-i)(n-i-1)}{2} \right) - S.$$

Proof: Necessity. Assume $d_3 < \sum_{i=1}^{n-2} \left(\ell_i \frac{(n-i)(n-i-1)}{2} \right) - S$,

that is, assume $d_3 \neq \sum_{i=1}^{n-2} \left(\ell_i \frac{(n-i)(n-i-1)}{2} \right) - S$ (by theorem 3.2.5.).

There must be a combination of three variables x_{j_1}, x_{j_2} and x_{j_3}, $1 \leq j_1 < j_2 < j_3 \leq n$, that contribute $b < \ell_{j_1} - \delta_{j_1 j_2 j_3}$ to d_3.

Then either of the two following cases exist.

case (i): $\delta_{j_1 j_2 j_3} = 0$. Then, $S_{j_1 j_2 j_3} = M(G_{x_{j_1} \bar{x}_{j_2} \bar{x}_{j_3}})$ $\leq r_{j_1 j_2 j_3} = M(G_{\bar{x}_{j_i} x_{j_2} x_{j_3}})$.

Hence, $G_{x_{j_1} \bar{x}_{j_2} \bar{x}_{j_3}} \not\supset G_{\bar{x}_{j_1} x_{j_2} x_{j_3}}$. Also, since variables x_{j_1}, x_{j_2}, and x_{j_3} contribute $b < \ell_{j_1} - \delta_{j_1 j_2 j_3} = \ell_{j_1}$ to d_3, there exists a vertex in $G_{x_{j_1} \bar{x}_{j_2} \bar{x}_{j_3}}$ that is not in $G_{\bar{x}_{j_1} x_{j_2} x_{j_3}}$.

Hence, $G_{x_{j_1} \bar{x}_{j_2} \bar{x}_{j_3}} \not\subseteq G_{\bar{x}_{j_1} x_{j_2} x_{j_3}}$. Therefore, G is not 3-monotonic, a contradiction.

case (ii): $\delta_{j_1 j_2 j_3} > 0$. Then, $S_{j_1 j_2 j_3} =$

$M(G_{x_{j_1}\overline{x}_{j_2}\overline{x}_{j_3}}) > r_{j_1 j_2 j_3} = M(G_{\overline{x}_{j_1} x_{j_2} x_{j_3}})$.

Hence, $G_{x_{j_1}\overline{x}_{j_2}\overline{x}_{j_3}} \not\subseteq G_{\overline{x}_{j_1} x_{j_2} x_{j_3}}$. Also, since variables x_{j_1}, x_{j_2}, and x_{j_3} contribute $b < \ell_{j_1} - \delta_{j_1 j_2 j_3}$ to d_3, there exists a vertex in $G_{\overline{x}_{j_1} x_{j_2} x_{j_3}}$ that is not in $G_{x_{j_1}\overline{x}_{j_2}\overline{x}_{j_3}}$. Hence, $G_{x_{j_1}\overline{x}_{j_2}\overline{x}_{j_3}} \not\supseteq G_{\overline{x}_{j_1} x_{j_2} x_{j_3}}$. Therefore, G is not 3-monotonic, a contradiction.

Sufficiency. Assume G is not 3-monotonic. There must exist a combination of three variables x_{j_1}, x_{j_2} and x_{j_3}, $1 \le j_1 < j_2 < j_3 \le n$, where $G_{x_{j_1}\overline{x}_{j_2}\overline{x}_{j_3}}$ contains a vertex that is not in $G_{\overline{x}_{j_1} x_{j_2} x_{j_3}}$, and $G_{\overline{x}_{j_1} x_{j_2} x_{j_3}}$ contains a vertex that is not in $G_{x_{j_1}\overline{x}_{j_2}\overline{x}_{j_3}}$. Since G is 2-monotonic and has Chow parameters $(M, \ell_1, \ldots, \ell_n)$ that satisfy the relations $\ell_1 \le \ell_2 \le \cdots \le \ell_n \le M/2$, variables x_{j_1}, x_{j_2}, and x_{j_3} must contribute $p = M(G_{x_{j_1} x_{j_2} \overline{x}_{j_3}}) + M(G_{x_{j_1}\overline{x}_{j_2} x_{j_3}}) + M(G_{x_{j_1} x_{j_2} x_{j_3}})$ to d_3. Also, $\ell_{j_1} = p + M(G_{x_{j_1}\overline{x}_{j_2}\overline{x}_{j_3}})$. Then one of the two following cases must exist.

case (i): $\delta_{j_1j_2j_3} = 0$. Then, $S_{j_1j_2j_3} = M(G_{x_{j_1}\bar{x}_{j_2}\bar{x}_{j_3}})$ $\leq r_{j_1j_2j_3} = M(G_{\bar{x}_{j_1}x_{j_2}x_{j_3}})$. Since $G_{x_{j_1}\bar{x}_{j_2}\bar{x}_{j_3}}$ contains a vertex that is not in $G_{\bar{x}_{j_1}x_{j_2}x_{j_3}}$, variables x_{j_1}, x_{j_2} and x_{j_3} can, at most, contribute $p + (M(G_{x_{j_1}\bar{x}_{j_2}\bar{x}_{j_3}}) - 1) =$ $\ell_{j_1} - 1 = \ell_{j_1} - \delta_{j_1j_2j_3} - 1$ to d_3. Hence, using theorem 3.2.5., $d_3 < \sum_{i=1}^{n-2}\left(\ell_i \frac{(n-i)(n-i-1)}{2}\right) - S$, a contradiction.

case (ii): $\delta_{j_1j_2j_3} > 0$. Then, $S_{j_1j_2j_3} =$ $M(G_{x_{j_1}\bar{x}_{j_2}\bar{x}_{j_3}}) > r_{j_1j_2j_3} = M(G_{\bar{x}_{j_1}x_{j_2}x_{j_3}})$. Since $G_{\bar{x}_{j_1}x_{j_2}x_{j_3}}$ contains a vertex that is not in $G_{x_{j_1}\bar{x}_{j_2}\bar{x}_{j_3}}$, variables x_{j_1}, x_{j_2}, and x_{j_3} can, at most, contribute $p + (M(G_{\bar{x}_{j_1}x_{j_2}x_{j_3}}) - 1)$

$$= p + M(G_{x_{j_1}\bar{x}_{j_2}\bar{x}_{j_3}}) - (M(G_{x_{j_1}\bar{x}_{j_2}\bar{x}_{j_3}}) - M(G_{\bar{x}_{j_1}x_{j_2}x_{j_3}})) - 1$$

$$= \ell_{j_1} - \delta_{j_1j_2j_3} - 1$$

to d_3. Hence, using theorem 3.2.5., $d_3 < \sum_{i=1}^{n-2}\left(\ell_i \frac{(n-i)(n-i-1)}{2}\right)$ $- S$, a contradiction. Q.E.D.

Corollary 3.2.6-1. The Boolean function $G(x_1, \ldots, x_n)$ is 3-monotonic iff $d_1 = \sum_{i=1}^{n} \ell_i$, $d_2 = \sum_{i=1}^{n-1} \ell_i (n-i)$, and $d_3 = \sum_{i=1}^{n-2} \left(\ell_i \frac{(n-i)(n-i-1)}{2}\right) - S$.

Theorem 3.2.7. For the Boolean function $G(x_1, \ldots, x_n)$,

$$d_r \leq \sum_{i=1}^{n-(r-1)} \ell_i \frac{(n-i)!}{[(n-i)-(r-1)]!(r-1)!}, \quad 1 \leq r < n.$$

Proof: Assume $d_r > \sum_{i=1}^{n-(r-1)} \ell_i \frac{(n-i)!}{[(n-i)-(r-1)]!(r-1)!}$.

There must exist a set of r variables $x_{j_1}, x_{j_2}, \ldots, x_{j_r}$, $1 \leq j_1 < j_2 < \ldots < j_r \leq n$, that contribute $b > \ell_{j_1}$ to d_r. That is, there are b unique pairs of vertices in G where the k^{th} element, $1 \leq k \leq n$, of each pair of vertices differ iff $K \in \{j_1, j_2, \ldots, j_r\}$. Hence, there are b vertices in G where the j_1^{th} element is 1.

Then, $M(G_{x_{j_1}}) = b > \ell_{j_1}$.

But, $M(G_{x_{j_1}}) = \ell_{j_1}$, a contradiction. Q.E.D.

Theorem 3.2.8. If the Boolean function $G(x_1, \ldots, x_n)$ is $(r-1)$-monotonic, $3 \leq r < n$, then G is r-monotonic if

$$d_r = \sum_{i=1}^{n-(r-1)} \ell_i \frac{(n-i)!}{[(n-i)-(r-1)]!(r-1)!}$$

Proof: Assume G is not r-monotonic.

There must exist a set of r variables $x_{j_1}, x_{j_2}, \ldots, x_{j_r}$, $1 \le j_1 < j_2 < \ldots < j_r \le n$, where the reduced functions $G_{y_1 \cdots y_r}$ and $G_{\bar{y}_1 \cdots \bar{y}_r}$, $y_{j_i} \in \{x_{j_i}, \bar{x}_{j_i}\}$, $i=1,\ldots,r$, are not comparable. Then $G_{y_{j_1} \cdots y_{j_r}}$ contains a vertex X_1 that is not in $G_{\bar{y}_{j_1} \cdots \bar{y}_{j_r}}$ and $G_{\bar{y}_{j_1} \cdots \bar{y}_{j_r}}$ contains a vertex X_2 that is not in $G_{y_{j_1} \cdots y_{j_r}}$. Hence, vertex X_1 or X_2 has its j_1th element equal to 1. Since $M(G_{x_{j_1}}) = \ell_{j_1}$, variables $x_{j_1}, \ldots, x_{j_r}$ contribute, at most, $(\ell_{j_1} - 1)$ to d_r. Using theorem 3.2.7., then

$$d_r < \sum_{i=1}^{n-(r-1)} \ell_i \frac{(n-i)!}{[(n-i)-(r-1)]!(r-1)!}$$

a contradiction. Q.E.D.

Corollary 3.2.8-1. Let the Boolean function $F(x_1, \ldots, x_n)$ be (r-1)-monotonic, $3 \le r < n$; F is r-monotonic

$$\text{if } d_r = \sum_{i=1}^{n-(r-1)} \ell_i \frac{(n-i)!}{[(n-i)-(r-1)]!\,(r-1)!}$$

Theorem 3.2.9. The Boolean function $G(x_1, \ldots, x_n)$ is completely monotonic if $M(M-1)/2 = \sum_{r=1}^{n} \ell_r 2^{n-r}$.

Proof: The number of distinct combinations of two vertices of G is equal to the total number of individual Hamming distances of G. That is,

$$M(M-1)/2 = \sum_{r=1}^{n} dr.$$

If $d_1 = \sum_{i=1}^{n} \ell_i$, then G is 1-monotonic (by theorem 3.2.2) and d_1 is maximum (by theorem 3.2.1).

If G is 1-monotonic and $d_2 = \sum_{i=1}^{n-1} \ell_i (n-i)$, then g is 2-monotonic (by theorem 3.2.4) and d_2 is maximum (by theorem 3.2.3.).

If G is (r-1)-monotonic, $3 \leq r < n$, and

$$d_r = \sum_{i=1}^{n-(r-1)} \ell_i \frac{(n-i)!}{[(n-i)-(r-1)]!\,(r-1)!}$$

then G is r-monotonic (by theorem 3.2.8.) and d_r is maximum (by theorem 3.2.7.).

Hence, if

$$\frac{M(M-1)}{2} = \sum_{r=1}^{n} d_r = \sum_{r=1}^{n} \sum_{i=1}^{n-(r-1)} \ell_i \frac{(n-i)!}{[(n-i)-(r-1)]!\,(r-1)!}$$

then d_r is maximum, $r=1, \ldots, n$; therefore, G is n-monotonic or completely monotonic.

But, $$\sum_{r=1}^{n} \sum_{i=1}^{n-(r-1)} \ell_i \frac{(n-i)!}{[(n-i)-(r-1)]!\,(r-1)!} =$$

$$= \sum_{i=1}^{n} \ell_i + \sum_{i=1}^{n-1} \ell_i (n-i) + \sum_{i=1}^{n-2} \ell_i \frac{(n-i)(n-i-1)}{2} + \ldots$$

$$= \sum_{r=1}^{n} \ell_r \sum_{i=0}^{n-r} \frac{(n-r)!}{(n-r-i)!\,i!}$$

$$= \sum_{r=1}^{n} \ell_r 2^{n-r}$$ Q.E.D.

Corollary 3.2.9-1. The Boolean function $F(x_1, \ldots, x_n)$ is completely monotonic if $M(M-1)/2 =$

$$\sum_{r=1}^{n} \ell_r 2^{n-r}.$$

3.3 Dual Comparability, Monotonicity, and the largest distance D

For a Boolean function $F(x_1, \ldots, x_n)$, D is the largest Hamming distance between any two vertices of F. That is, if F has a distance vector $d=(d_1, \ldots, d_n)$, then D is the largest value of i where $d_i \neq 0$; hence, $1 \leq D \leq n$.

Theorem 3.3.1. If a Boolean function $F(x_1, \ldots, x_n)$ is $[D/2]$ -monotonic, where $[D/2]$ is the largest integer $\leq D/2$, then F is 2-asummable.

Proof: Assume F is 2-summable.

There must exist two vertices $U \in F$ and $V \in F$ and two vertices $Y \in \overline{F}$ and $Z \in \overline{F}$ such that

$$U + V = Y + Z$$

For $U = U_1U_2 \cdots U_n$, $V = V_1V_2 \cdots V_n$, $Y = Y_1Y_2 \cdots Y_n$, and $Z = Z_1Z_2 \cdots Z_n$, then $U_i + V_i = Y_i + Z_i$, $i=1, \ldots, n$.

If $V_i = U_i$, then $Y_i = Z_i = U_i = V_i$.

If $V_i = \overline{U}_i$, then either $Y_i = U_i$ and $Z_i = V_i = \overline{U}_i$ or $Y_i = V_i = \overline{U}_i$ and $Z_i = U_i$.

Let the Hamming distance between U and V be $D' \leq D$.

Without loss of generality, relabel the subscripts so that $V_i = \overline{U}_i$, $i=1, \ldots, D'$, and $Y_i = U_i$, $i=1, \ldots, k$ where $k \leq D'$.

Then, $$U = U_1 \cdots U_{k\;k+1} \cdots U_{D'}U_{D'+1} \cdots U_D \cdots U_n,$$

$$V = \overline{U}_1 \cdots \overline{U}_k\overline{U}_{k+1} \cdots \overline{U}_{D'}U_{D'+1} \cdots U_D \cdots U_n,$$

$$Y = U_1 \cdots U_k\overline{U}_{k+1} \cdots \overline{U}_{D'}U_{D'+1} \cdots U_D \cdots U_n,$$

and $$Z = \overline{U}_1 \cdots \overline{U}_kU_{k+1} \cdots U_{D'}U_{D'+1} \cdots U_D \cdots U_n.$$

Either one or both of the following cases exist.

case (i): $K \leq [D/2]$. Since F is $[D/2]$ -monotonic, it is necessary that $F_{U_1 \ldots U_k} : F_{\overline{U}_1 \ldots \overline{U}_k}$; also, since vertices U and V are in F, then either

$$\text{vertex } U_1 \cdots U_k\overline{U}_{k+1} \cdots \overline{U}_{D'}U_{D'+1} \cdots U_D \cdots U_n = Y \in F.$$

$$\text{or vertex } \overline{U}_1 \cdots \overline{U}_kU_{k+1} \cdots U_{D'}U_{D'+1} \cdots U_D \cdots U_n = Z \in F.$$

But, vertex $Y \in \overline{F}$ and vertex $Z \in \overline{F}$.

Hence, F is not $[D/2]$ -monotonic, a contradiction.

case (ii): $D'-k \leq [D/2]$. Since F is $[D/2]$ -monotonic, it is necessary that $F_{U_{k+1} \cdots U_{D'}} : F_{\overline{U}_{k+1} \cdots \overline{U}_{D'}}$; also, since vertices U and V are in F, then either

$$\text{vertex } U_1 \cdots U_k\overline{U}_{k+1} \cdots \overline{U}_{D'}U_{D'+1} \cdots U_D \cdots U_n = Y \in F.$$

or vertex $\bar{U}_1 \cdots \bar{U}_k U_{k+1} \cdots U_{D'} U_{D'+1} \cdots U_D \cdots U_n = Z \in F$.

But, vertex $Y \in \bar{F}$ and vertex $Z \in \bar{F}$. Hence, F is not $[D/2]$ -monotonic, a contradiction. Q.E.D.

Theorem 3.3.2. A Boolean function $F(x_1, \ldots, x_n)$ is completely monotonic iff F is $[D/2]$ -monotonic, where $[D/2]$ is the largest integer $\leq D/2$.

Proof: Necessity. Assume F is $[D/2]$-monotonic. Then, by theorem 3.3.1., F is 2-asummable.

By Proposition 2.2.3.1., F is completely monotonic.

Sufficiency. Assume F is completely monotonic. Then, by the definition of complete monotonicity, F is n-monotonic. Since $n \geq D \geq [D/2]$, then F is $[D/2]$ -monotonic.

Q.E.D.

Theorem 3.3.3. For a Boolean function $F(x_1, \ldots, x_n)$ composed of M vertices, where $M<2^{n-1}$, F is dual-comparable iff $D<n$.

Proof: Necessity. Assume $D=n$ (i.e., $D \not< n$). Since $D=n$, F must contain a vertex $X_1 = x_{11}x_{12}\cdots x_{1n}$ and a vertex $\bar{X}_1 = \bar{x}_{11}\bar{x}_{12}\cdots\bar{x}_{1n}$. Also, $M(F^D(x_1, \ldots, x_n)) = M(\bar{F}(\bar{x}_1, \ldots, \bar{x}_n)) = M(\bar{F}(x_1, \ldots, x_n)) = 2^n - M$.

Since $M(F(x_1, \ldots, x_n)) = M<2^{n-1}$, then $M(F^D(x_1, \ldots, x_n))$ $= 2^n-M>2^{n-1}>M$. Hence, $F(x_1, \ldots, x_n) \not\geq F^D(x_1, \ldots, x_n)$. Since vertex $X_1 = x_{11}x_{12}\cdots x_{1n} \in F(x_1, \ldots, x_n)$ and vertex $\bar{X}_1 = \bar{x}_{11}\bar{x}_{12}\cdots\bar{x}_{1n} \in F(x_1, \ldots, x_n)$, then vertex $\bar{X}_1 =$ $\bar{x}_{11}\bar{x}_{12}\cdots\bar{x}_{1n} \in F(\bar{x}_1, \ldots, \bar{x}_n)$ and vertex $X_1 = x_{11}x_{12}\cdots x_{1n}$ $\in F(\bar{x}_1, \ldots, \bar{x}_n)$. But, if vertex $\bar{X}_1 \in F(\bar{x}_1, \ldots, \bar{x}_n)$ and vertex $X_1 \in F(\bar{x}_1, \ldots, \bar{x}_n)$, then vertex $X_1 \notin \bar{F}(\bar{x}_1, \ldots, \bar{x}_n)$ $= F^D(x_1, \ldots, x_n)$ and vertex $\bar{X}_1 \notin \bar{F}(\bar{x}_1, \ldots, \bar{x}_n) = F^D(x_1,$ $\ldots, x_n)$. Hence, $F \not\leq F^D$. Therefore, F is not dual-comparable, a contradiction.

Sufficiency. Assume F is not dual-comparable. Since $M(F(x_1, \ldots, x_n)) = M<2^{n-1}<2^n-M = M(F^D(x_1, \ldots, x_n))$, then $F(x_1, \ldots, x_n) \not\geq F^D(x_1, \ldots, x_n)$. Also, since F is not dual-comparable, there must exist a vertex $X_1 = x_{11}x_{12}$ $\ldots x_{1n} \in F$ such that $X_1 \notin F^D$. But, if $X_1 \notin F^D = \bar{F}(\bar{x}_1, \ldots, \bar{x}_n)$, then $X_1 = x_{11}x_{12}\cdots x_{1n} \in F(\bar{x}_1, \ldots, \bar{x}_n)$; hence, if $X_1 =$

$x_{11}x_{12}\ldots x_{1n} \in F(\bar{x}_1, \ldots, \bar{x}_n)$, then $\bar{X}_1 = \bar{x}_{11}\bar{x}_{12}\ldots\bar{x}_{1n}$ $\in F(x_1, \ldots, x_n)$. But, if $X_1 \in F$ and $\bar{X}_1 \in F$, then D=n, a contradiction. Q.E.D.

Theorem 3.3.4. For a Boolean function $F(x_1, \ldots, x_n)$ composed of M vertices, where $M = 2^{n-1}$, F is self-dual iff D<n.

Proof: Necessity. Assume D=n (i.e., $D \not< n$). Since d=n, then F must contain two vertices $X_1 = x_{11}x_{12}\ldots x_{1n}$ and $\bar{X}_1 = \bar{x}_{11}\bar{x}_{12}\ldots\bar{x}_n$. Also, since F is self-dual, then $X_1 \in F^D = \bar{F}(\bar{x}_1, \ldots, \bar{x}_n)$; if $X_1 \in \bar{F}(\bar{x}_1, \ldots, \bar{x}_n)$, then $\bar{X}_1 \in \bar{F}(x_1, \ldots, x_n)$. But, $\bar{X}_1 \in F(x_1, \ldots, x_n)$ or $\bar{X}_1 \notin \bar{F}(x_1, \ldots, x_n)$. Hence, F is not self-dual, a contradiction.

Sufficiency. Assume F is not self-dual. Since F is not self-dual, there exists a vertex $X_1 = x_{11}x_{12}$ $\ldots x_{1n} \in F$ where $X_1 \notin F^D = \bar{F}(\bar{x}_1, \ldots, \bar{x}_n)$. If vertex $X_1 \notin \bar{F}(\bar{x}_1, \ldots, \bar{x}_n)$, then vertex $X_1 \in F(\bar{x}_1, \ldots, \bar{x}_n)$; if vertex $X_1 = x_{11}x_{12}\cdots x_{1n} \in F(\bar{x}_1, \ldots, \bar{x}_n)$, then vertex

$\bar{X}_1 = \bar{x}_{11}\bar{x}_{12}\cdots\bar{x}_{1n} \in F(x_1, \ldots, x_n)$. But, if vertex $X_1 \in F$ and vertex $\bar{X}_1 \in F$, then $D=n$, a contradiction. Q.E.D.

Theorem 3.3.5. A Boolean function $F(x_1, \ldots, x_n)$ is composed of M vertices; if $M \leq 2^{D-1}$, then F is not completely monotonic.

Proof: Assume F is completely monotonic. By theorem 2.3.3.1., F is 2-asummable. Also, F must contain two vertices U and V that have a Hamming distance of D. Relabel the subscripts so that

$$U = U_1U_2 \cdots U_DU_{D+1} \cdots U_n$$

and

$$V = \bar{U}_1\bar{U}_2 \cdots \bar{U}_DU_{D+1} \cdots U_n$$

But, there are exactly $(2^{D-1}-1)$ unique pairs of vertices $Y = Y_1Y_2\cdots Y_DU_{D+1}\cdots U_n$ and $Z = \bar{Y}_1\bar{Y}_2\cdots\bar{Y}_DU_{D+1}\cdots U_n$, excluding U and V, such that $U + V = Y + Z$. Since F is 2-asummable, F must contain at least one vertex of each of these $(2^{D-1}-1)$ pairs of vertices, Y and Z. Therefore, $M(F) = 2 + 2^{D-1} - 1$; that is, $M(F) \not\leq 2^{D-1}$, a contradiction. Q.E.D.

3.4 Methods to Determine the Parameter D

For a Boolean function $F(x_1, \ldots, x_n)$ where M(F)

≥ 2, there are several ways to calculate the parameter D which is the largest Hamming distance between two vertices in F. One way to calculate D is to determine the distance vector $d = (d_1, \ldots, d_n)$ of F; D is the largest value of i where $d_i \neq 0$ and $1 \leq i \leq n$. Another way to determine D is to consider all combinations of two vertices of F and record the largest hamming distance found between two vertices.

A relatively simple way to calculate D is as follows. Write $F(x_1, \ldots, x_n)$ in minimum sum-of-product form so that F is the sum of essential prime implicants. If a variable x_i, $1 \leq i \leq n$, occurs in F in negated and un-negated forms, then F is not 1-monotonic (by Corollary 2.2.1.4-1.); if F is unate, the following method, to calculate D, can be used. Form a largest non-repetitive set consisting of a combination of not more than two sets of the independent variables of the essential prime implicants; the number of elements in this largest set, is the value of D.

Example. $F(x_1, \ldots, x_6) = x_1x_2x_5x_6 + x_1x_3x_4x_5x_6 + x_2x_4x_5x_6 + x_1x_2x_3x_4x_5 + x_1x_2x_3x_4x_6$. The five sets of variables that are independent of the five essential prime implicants of the 1-monotonic Boolean function $F(x_1, \ldots, x_n)$ are: $\{x_3, x_4\}$, $\{x_2\}$, $\{x_1, x_3\}$, $\{x_6\}$, and $\{x_5\}$. The largest non-repetitive

sets consisting of any combination of two of these independent sets are: $\{x_2,x_3,x_4\}$, $\{x_1,x_3,x_4\}$, $\{x_3,x_4,x_6\}$, $\{x_3,x_4,x_5\}$, $\{x_1,x_2,x_3\}$, $\{x_1,x_3,x_6\}$, and $\{x_1,x_3,x_5\}$. The size of a largest set is 3; hence, D=3.

3.5 Monotonicity and the Distance Vectors $\bar{d}$ and d'

A Boolean n-space B^n contains 2^n vertices. If a Boolean function $F(x_1, \ldots, x_n)$ has Chow parameters $(M, C_1, \ldots, C_n)$, a canonical vector $\ell = (\ell_1, \ldots, \ell_n)$, and a distance vector $d = (d_1, \ldots, d_n)$, then the Boolean function $\bar{F}(x_1, \ldots, x_n)$ has Chow parameters $(M(\bar{F}), C_1(\bar{F}), \ldots, C_n(\bar{F}))$ or $(2^n-M, 2^{n-1}-C_1, \ldots, 2^{n-1}-C_n)$, a canonical vector $\bar{\ell} = (\bar{\ell}_1, \ldots, \bar{\ell}_n)$, and a distance vector $\bar{d} = (\bar{d}_1, \ldots, \bar{d}_n)$. For all i and j, $1 \leq i < j \leq n$, $\bar{\ell}_i = 2^{n-1}-M+\ell_i \leq \bar{\ell}_j = 2^{n-1} - M+ \ell_j$ because $\ell_i \leq \ell_j$; also, $\bar{\ell}_i = 2^{n-1}-(M-\ell_i) = 2^{n-1}-M+\ell_i$ and not $2^{n-1}-\ell_i$, $i=1, \ldots, n$, because $2^{n-1}-M+\ell_i \leq 2^{n-1}-\ell_i$ or $\ell_i \leq M/2 \leq M-\ell_i$. Since $-M+\ell_n \leq -M/2$, $\bar{\ell}_n = 2^{n-1}-M+\ell_n \leq M(\bar{F})/2 = \frac{2^n-M}{2}$. Hence, vector $\bar{\ell} = (\bar{\ell}_1, \ldots, \bar{\ell}_n)$ satisfies the relations $\bar{\ell}_1 \leq \bar{\ell}_2 \leq \cdots \leq \bar{\ell}_n \leq \frac{2^n-M}{2}$.

The number of combinations of n things r at a time is defined to be $\binom{n}{r} = \frac{n!}{(n-r)!r!}$. For A Boolean n-space, composed of 2^n vertices, there are $\binom{2^n}{2} = \frac{2^n(2^n-1)}{2}$ combinations of two vertices. From the set of n variables, the number of distinct ways of selecting r variables, $1 \leq r \leq n$, is $\binom{n}{r} = \frac{n!}{(n-r)!r!}$ There are 2^{n-r} vertices, in a n-cube, that are associated with each vertex in a r-cube; also, the

number of combinations of two vertices, in an r-cube, that have a Hamming distance of r is $\frac{2^r}{2} = 2^{r-1}$. Hence, for a Boolean n-space, the total number of combinations of two vertices that have a Hamming distance of r, $1 \leq r \leq n$, is $\binom{n}{r} 2^{n-r} 2^{r-1} = \frac{n!}{(n-r)!r!} 2^{n-1}$. In a Boolean n-space, the total number of combinations of two vertices that have a Hamming distance r, r=1, ..., n, equals the number of combinations of two vertices; hence, it should be true that

$$\sum_{r=1}^{n} \binom{n}{r} 2^{n-1} = \frac{2^n(2^n-1)}{2} .$$

Consider,

$$\text{L.S.} = \sum_{r=1}^{n} \binom{n}{r} 2^{n-1} = 2^{n-1} \sum_{r=1}^{n} \binom{n}{r} =$$

$$2^{n-1}(2^n-1) = \frac{2^n(2^n-1)}{2} = \text{R.S.}$$

Since $M(F) = M$ and $M(\overline{F}) = 2^n-M$, there are $M(2^n-M)$ combinations of two vertices where each combination of two vertices consists of one vertex in F and one vertex in $\overline{F}$. Let $d' = (d_1', \ldots, d_n')$ be the distance vector to record the Hamming distances of all combinations of two vertices where each combination of two vertices has one vertex in F and one vertex in $\overline{F}$. Then,

$$\sum_{i=1}^{n} d_i + \sum_{i=1}^{n} \overline{d}_i + \sum_{i=1}^{n} d_i' = \frac{2^n(2^n-1)}{2}.$$

Since,

$$\sum_{i=1}^{n} d_i = \frac{M(M-1)}{2}, \quad \sum_{i=1}^{n} \overline{d}_i = \frac{(2^n-M)(2^n-M-1)}{2},$$

and $\sum_{i=1}^{n} d_i' = M(2^n-M)$, then it should be true that

$$\frac{M(M-1)}{2} + \frac{(2^n-M)(2^n-M-1)}{2} + M(2^n-M) = \frac{2^n(2^n-1)}{2}.$$

Consider,

$$\text{L.S.} = \frac{M(M-1)}{2} + \frac{(2^n-M)(2^n-M-1)}{2} + M(2^n-M)$$

$$= \frac{M^2}{2} - \frac{M}{2} + 2^{2n-1} - M2^n - 2^{n-1} + \frac{M^2}{2} + \frac{M}{2} + M2^n - M^2$$

$$= 2^{2n-1} - 2^{n-1}$$

$$= \frac{2^n(2^n-1)}{2} = \text{R.S.}$$

Theorem 3.5.1. The Boolean function $F(x_1, \ldots, x_n)$ is 1-monotonic iff $\bar{d}_1 = n(2^{n-1}-M) + \sum_{i=1}^{n} \ell_i$.

Proof: Consider

$$\bar{d}_1 = n(2^{n-1}-M) + \sum_{i=1}^{n} \ell_i = \sum_{i=1}^{n} (2^{n-1}-M+\ell_i)$$

$$= \sum_{i=1}^{n} \bar{\ell}_i .$$

Using corollary 3.2.2-1., $\bar{F}$ is 1-monotonic iff $\bar{d}_1 = \sum_{i=1}^{n} \bar{\ell}_i$.

Hence, since F is 1-monotonic iff $\bar{F}$ is 1-monotonic, then F is 1-monotonic iff $\bar{d}_1 = n(2^{n-1}-M) + \sum_{i=1}^{n} \ell_i$.

Corollary 3.5.1-1. If the Boolean function $F(x_1, \ldots, x_n)$ is 1-monotonic, $\bar{d}_1 = n(2^{n-1}-M) + d_1$.

Proof: Assume F is 1-monotonic.

Then, $d_1 = \sum_{i=1}^{n} \ell_i$ (by corollary 3.2.2-1). By theorem 3.5.1.,

$\bar{d}_1 = n(2^{n-1}-M) + \sum_{i=1}^{n} \ell_i = n(2^{n-1}-M) + d_1$. Q.E.D.

Corollary 3.5.1-2. If $F(x_1, \ldots, x_n)$ is 1-monotonic and $M(F) = 2^{n-1}$, then $\bar{d}_1 = d_1$.

Proof: This follows from corollary 3.5.1-1.

Theorem 3.5.2. For a 1-monotonic function F, F is 2-monotonic iff $\bar{d}_2 = \frac{n(n-1)}{2}(2^{n-1}-M) + \sum_{i=1}^{n-1} \ell_i (n-i)$.

Proof: Since $\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}$, then

$$\bar{d}_2 = \frac{n(n-1)}{2}(2^{n-1}-M) + \sum_{i=1}^{n-1} \ell_i (n-i) =$$

$$= \sum_{i=1}^{n-1} \left[(2^{n-1}-M+\ell_i)(n-i)\right] = \sum_{i=1}^{n-1} \bar{\ell}_i (n-i).$$

$\bar{F}$ is 1-monotonic because F is 1-monotonic. Using theorem 3.2.4., $\bar{F}$ is 2-monotonic iff $\bar{d}_2 = \sum_{i=1}^{n-1} \bar{\ell}_i (n-i)$. Since F is 2-monotonic iff $\bar{F}$ is 2-monotonic, then F is 2-monotonic iff $\bar{d}_2 = \frac{n(n-1)}{2}(2^{n-1}-M) + \sum_{i=1}^{n-1} \ell_i (n-i)$. Q.E.D.

Corollary 3.5.2-1. The Boolean function $F(x_1,\ldots,x_n)$ is 2-monotonic iff $\bar{d}_1 = n(2^{n-1}-M) + \sum_{i=1}^{n} \ell_i$ and $\bar{d}_2 = \frac{n(n-1)}{2}(2^{n-1}-M) + \sum_{i=1}^{n-1} \ell_i (n-i)$.

Proof: By theorem 3.5.1, F is 1-monotonic iff $\bar{d}_1 = n(2^{n-1}-M) + \sum_{i=1}^{n} \ell_i$. By theorem 3.5.2., F is 2-monotonic iff

F is 1-monotonic and $\bar{d}_2 = \frac{n(n-1)}{2}(2^{n-1}-M) + \sum_{i=1}^{n-1} \ell_i (n-i)$.

Q.E.D.

Corollary 3.5.2-2. If the Boolean function $F(x_1, \ldots, x_n)$ is 2-monotonic, then $\bar{d}_1 = n(2^{n-1}-M) + d_1$ and $\bar{d}_2 = \frac{n(n-1)}{2}(2^{n-1}-M) + d_2$.

Proof: If F is 2-monotonic, then $d_1 = \sum_{i=1}^{n} \ell_i$ and $d_2 = \sum_{i=1}^{n-1} \ell_i (n-i)$ (by corollary 3.2.4-2). Hence, using corollary 3.5.2-1., $\bar{d}_1 = n(2^{n-1}-M) + d_1$ and $\bar{d}_2 = \frac{n(n-1)}{2}(2^{n-1}-M) + d_2$.

Q.E.D.

Corollary 3.5.2-3. If the Boolean function $F(x_1, \ldots, x_n)$ is 2-monotonic and $M(F) = 2^{n-1}$, then $\bar{d}_1 = d_1$ and $\bar{d}_2 = d_2$.

Proof: This follows from corollary 3.5.2-2.

Theorem 3.5.3. For a (r-1)-monotonic function F, $3 \le r < n$; F is r-monotonic if

$$\bar{d}_r = \sum_{i=1}^{n-(r-1)} \left[(2^{n-1}-M+\ell_i) \binom{n-i}{r-1} \right].$$

Proof: Assume that

$$\bar{d}_r = \sum_{i=1}^{n-(r-1)} \left[(2^n - M + \ell_i) \binom{n-i}{r-1} \right]$$

$$= \sum_{i=1}^{n-(r-1)} \left[\bar{\ell}_i \frac{(n-i)!}{[(n-i)-(r-1)]!\,(r-1)!} \right].$$

Since F is (r-1)-monotonic, then $\bar{F}$ is (r-1)-monotonic.

Using corollary 3.2.8-1., $\bar{F}$ is r-monotonic, hence, F is r-monotonic. Q.E.D.

Theorem 3.5.4. The Boolean function $F(x_1, \ldots, x_n)$ is completely monotonic if

$$\frac{(2^n-M)(2^n-M-1)}{2} = \sum_{r=1}^{n} (2^{n-1}-M+\ell_r)\, 2^{n-r}$$

Proof: Assume that

$$\frac{(2^n-M)(2^n-M-1)}{2} = \frac{M(\bar{F})[M(\bar{F})-1]}{2} =$$

$$= \sum_{r=1}^{n} (2^{n-1}-M+\ell_r) \quad 2^{n-r} = \sum_{r=1}^{n} \bar{\ell}_r 2^{n-r}.$$

Using corollary 3.2.9-1., $\bar{F}$ is completely monotonic; hence, F is completely monotonic. Q.E.D.

Theorem 3.5.5. The Boolean function $F(x_1, \ldots, x_n)$ is 1-monotonic iff $d_1' = nM - 2 \sum_{i=1}^{n} \ell_i$.

Proof: Consider,

$$d_1' = nM-2 \sum_{i=1}^{n} \ell_i = n2^{n-1} - \sum_{i=1}^{n} \ell_i - n\,(2^{n-1}-M) - \sum_{i=1}^{n} \ell_i$$

$$= \binom{n}{1} 2^{n-1} - \sum_{i=1}^{n} \ell_i - \sum_{i=1}^{n} (2^{n-1}-M+\ell_i)$$

$$= \binom{n}{1} 2^{n-1} - \sum_{i=1}^{n} \ell_i - \sum_{i=1}^{n} \mathcal{I}_i .$$

But, $d_1' = \binom{n}{1} \; 2^{n-1} - d_1 - \bar{d}_1$.

Hence, $d_1 + \bar{d}_1 = \sum_{i=1}^{n} \ell_i + \sum_{i=1}^{n} \ell_i$.

By theorem 3.2.1., $d_1 \leq \sum_{i=1}^{n} \ell_i$; using theorem 3.2.1, $\bar{d}_1 \leq \sum_{i=1}^{n} \mathcal{I}_i$.

Hence, $d_1' = nM - 2 \sum_{i=1}^{n} \ell_i$ iff $d_1 = \sum_{i=1}^{n} \ell_i$ and $\bar{d}_1 = \sum_{i=1}^{n} \mathcal{I}_i$.

By corollary 3.2.2-1., F is 1-monotonic iff $d_1 = \sum_{i=1}^{n} \ell_i$.

Therefore, F is 1-monotonic iff $d_1' = nM-2 \sum_{i=1}^{n} \ell_i$. Q.E.D.

Corollary 3.5.5-1. If the Boolean function $F(x_1, \ldots, x_n)$ is 1-monotonic, then $d_1' = nM - 2d_1$.

Proof: Since F is 1-monotonic, $d_1 = \sum_{i=1}^{n} \ell_i$ (by corollary 3.2.2-1).

By theorem 3.5.5., $d_1' = nM - 2 \sum_{i=1}^{n} \ell_i = nM - 2d_1$. Q.E.D.

Theorem 3.5.6. For a 1-monotonic function F,

F is 2-monotonic iff $d_2' = \frac{n(n-1)}{2} M - 2 \sum_{i=1}^{n-1} \ell_i (n-i)$.

Proof: Consider,

$$d_2' = \frac{n(n-1)}{2} M - 2 \sum_{i=1}^{n-1} \ell_i (n-i)$$

$$= \frac{n(n-1)}{2} 2^{n-1} - \sum_{i=1}^{n-1} \ell_i (n-i) - \frac{n(n-1)}{2} (2^{n-1} - M) - \sum_{i=1}^{n-1} \ell_i (n-i)$$

$$= \binom{n}{2} 2^{n-1} - \sum_{i=1}^{n-1} \ell_i (n-i) - \sum_{i=1}^{n-1} (2^{n-1} - M + \ell_i)(n-i)$$

$$= \binom{n}{2} 2^{n-1} - \sum_{i=1}^{n-1} \ell_i (n-i) - \sum_{i=1}^{n-1} \bar{\ell}_i (n-i).$$

But, $d_2' = \binom{n}{2} 2^{n-1} - d_2 - \bar{d}_2$.

Hence, $d_2 + \bar{d}_2 = \sum_{i=1}^{n-1} \ell_i (n-i) + \sum_{i=1}^{n-1} \bar{\ell}_i (n-i)$

By theorem 3.2.3, $d_2 \leq \sum_{i=1}^{n-1} \ell_i (n-i)$; using theorem 3.2.3.,

$$\bar{d}_2 \leq \sum_{i=1}^{n-1} \bar{\ell}_i (n-i).$$

Hence, $d_2' = \frac{n(n-1)}{2} M - 2 \sum_{i=1}^{n-1} \ell_i (n-i)$ iff $d_2 = \sum_{i=1}^{n-1} \ell_i (n-i)$

and $\bar{d}_2 = \sum_{i=1}^{n-1} \bar{\ell}_i (n-i)$,

Using theorem 3.2.4., F is 2-monotonic iff $d_2 = \sum_{i=1}^{n-1} \ell_i (n-i)$.

Therefore, F is 2-monotonic iff $d_2' = \frac{n(n-1)}{2} M - 2 \sum_{i=1}^{n-1} \ell_i (n-i)$

Q.E.D.

Corollary 3.5.6-1. If $F(x_1, \ldots, x_n)$ is 2-monotonic, then $d_2' = \frac{n(n-1)}{2} M - 2d_2$.

Proof: Since F is 2-monotonic, $d_2 = \sum_{i=1}^{n} \ell_i (n-i)$ (by corollary 3.2.4-2.).

By theorem 3.5.6., $d_2' = \frac{n(n-1)}{2} M - 2 \sum_{i=1}^{n-1} \ell_i (n-i) = \frac{n(n-1)}{2} M - 2d_2$.

Q.E.D.

Theorem 3.5.7. For a (r-1)-monotonic function F, $3 \leq r < n$, F is r-monotonic if

$$d_r' = \binom{n}{r} 2^{n-1} - \sum_{i=1}^{n-(r-1)} \left[(2^{n-1} - M + 2\ell_i) \binom{n-i}{r-1} \right].$$

Proof: Assume

$$d_r' = \binom{n}{r} 2^{n-1} - \sum_{i=1}^{n-(r-1)} \left[(2^{n-1} - M + 2\ell_i) \binom{n-i}{r-1} \right]$$

$$= \binom{n}{r} 2^{n-1} - \sum_{i=1}^{n-(r-1)} \ell_i \binom{n-i}{r-1} -$$

$$\sum_{i=1}^{n-(r-1)} \left[(2^{n-1} - M + \ell_i) \binom{n-i}{r-1} \right]$$

$$= \binom{n}{r} 2^{n-1} - \sum_{i=1}^{n-(r-1)} \ell_i \binom{n-i}{r-1} - \sum_{i=1}^{n-(r-1)} \bar{\ell}_i \binom{n-i}{r-1}.$$

But, $d_r' = \binom{n}{r} 2^{n-1} - d_r - \bar{d}_r$

Hence, $d_r + \bar{d}_r = \sum_{i=1}^{n-(r-1)} \ell_i \binom{n-i}{r-1} + \sum_{i=1}^{n-(r-1)} \bar{\ell}_i \binom{n-i}{r-1}$. By theorem 3.2.7,

$d_r \leq \sum_{i=1}^{n-(r-1)} \ell_i \binom{n-i}{r-1}$; using theorem 3.2.7.,

$$\bar{d}_r \leq \sum_{i=1}^{n-(r-1)} \bar{\ell}_i \binom{n-i}{r-1}.$$

Hence, $d_r' = \binom{n}{r} 2^{n-1} - \sum_{i=1}^{n-(r-1)} \left[(2^{n-1} - M + 2\ell_i) \binom{n-i}{r-1}\right]$ iff

$$d_r = \sum_{i=1}^{n-(r-1)} \ell_i \binom{n-i}{r-1}$$

and $\bar{d}_r = \sum_{i=1}^{n-(r-1)} \bar{\ell}_i \binom{n-i}{r-1}$. By corollary 3.2.8-1., F is r-monotonic if F is (r-1)-monotonic and $d_r = \sum_{i=1}^{n-(r-1)} \ell_i \binom{n-i}{r-1}$.

Therefore, if $d_r' = \binom{n}{r} 2^{n-1} - \sum_{i=1}^{n-(r-1)} \left[(2^{n-1} - M + 2\ell_i) \binom{n-i}{r-1}\right]$,

then F is r-monotonic. Q.E.D.

Theorem 3.5.8. F is completely monotonic if

$M(2^n - M) = \frac{2^n(2^n-1)}{2} - \sum_{r=1}^{n} (2^{n-1} - M + 2\ell_r) 2^{n-r}$.

Proof: Assume,

$$M(2^n - M) = \frac{2^n(2^n-1)}{2} - \sum_{r=1}^{n} (2^{n-1} - M + 2\ell_r) 2^{n-r}$$

$$= \frac{2^n(2^n-1)}{2} - \sum_{r=1}^{n} \ell_r 2^{n-r} - \sum_{r=1}^{n} (2^{n-1}-M+\ell_r)\ 2^{n-r}$$

But, $\sum_{i=1}^{n} d_i' = \binom{2^n}{2} - \sum_{i=1}^{n} d_i - \sum_{i=1}^{n} \bar{d}_i$

or $\quad M(2^n-M) = \frac{2^n(2^n-1)}{2} - \frac{M(M-1)}{2} - \frac{(2^n-M)(2^n-M-1)}{2}$

Hence,

$$\frac{M(M-1)}{2} + \frac{(2^n-M)(2^n-M-1)}{2} = \sum_{r=1}^{n} \ell_r 2^{n-r} + \sum_{r=1}^{n} (2^{n-1}-M+\ell_r)\,2^{n-r}.$$

By theorem 3.2.7., $d_r \leq \sum_{i=1}^{n-(r-1)} \ell_i \binom{n-i}{r-1}$ $1 \leq r \leq n$; then,

$$\sum_{r=1}^{n} d_r = \frac{M(M-1)}{2} \leq \sum_{r=1}^{n} \left[\sum_{i=1}^{n-(r-1)} \ell_i \binom{n-i}{r-1}\right] = \sum_{r=1}^{n} \ell_r 2^{n-r}.$$

Similarly, using theorem 3.2.7.,

$$\bar{d}_r \leq \sum_{i=1}^{n-(r-1)} \bar{\ell}_i \binom{n-i}{r-1} = \sum_{i=1}^{n-(r-1)} (2^{n-1}-M+\ell_i) \binom{n-i}{r-1}, \quad 1 \leq r \leq n.$$

Then,

$$\sum_{i=1}^{n} \bar{d}_r = \frac{(2^n-M)(2^n-M-1)}{2} \leq \sum_{r=1}^{n} \left[\sum_{i=1}^{n-(r-1)} (2^{n-1}-M+\ell_i) \binom{n-i}{r-1}\right]$$

$$= \sum_{r=1}^{n} (2^{n-1}-M+\ell_i)\ 2^{n-r}.$$

Hence,

$$\frac{M(M-1)}{2} = \sum_{r=1}^{n} \ell_r\ 2^{n-r}.$$

Therefore, by corollary 3.2.9-1., F is completely monotonic.

Q.E.D.

CHAPTER 4

PROPERTIES OF BOOLEAN FUNCTIONS AND THE MUTUAL DISTANCE TOTAL Q

4.1 The Mutual Distance Total

A Boolean function $F(x_1, \ldots, x_n)$ has Chow parameters $(M, C_1, \ldots, C_n)$. The mutual distance total Q of F is defined to be the sum of the Hamming distances of all combinations of two vertices of F. If $M(F)<2$, then the mutual distance distance total of F is not defined. The mutual distance total Q of F may be calculated using the Chow parameters and the formula

$$Q = \sum_{i=1}^{n} C_i \ (M-C_i).$$

That is, since F is composed of M vertices where the i^{th} element, $1 \leq i \leq n$, is 1 for C_i vertices and is 0 for $(M-C_i)$ vertices, there are exactly $(M-C_i)\ C_i$ combinations of two vertices of F such that each pair of vertices differ in the i^{th} element. Another method of determining the mutual distance total of F is by using the distance vector $d=(d_1, \ldots, d_n)$ of F and the formula

$$Q = \sum_{i=1}^{n} id_i.$$

If the M vertices of F are denoted by $X_i = x_{i1}x_{i2}\ldots x_{in}$, $i=1,\ldots,M$, then the mutual distance total Q of F may be calculated using the formula

$$Q = \frac{1}{2} \sum_{i=1}^{M} \sum_{j=1}^{n} x_{ij}(M-C_j) + (1-x_{ij})\, C_j \quad .$$

This method is valid since

$$Q = \frac{1}{2} \sum_{i=1}^{M} \sum_{j=1}^{n} x_{ij}(M-C_j) + (1-x_{ij})\, C_j$$

$$= \frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{M} x_{ij}(M-C_j) + (1-x_{ij})\, C_j$$

$$= \frac{1}{2} \sum_{j=1}^{n} (MC_j - C_j^{\,2} + MC_j - C_j^{\,2})$$

$$= \sum_{j=1}^{n} C_j\ (M-C_j).$$

Hence, for a vertex $X_i = x_{i1}x_{i2}\ldots x_{in} \in F$, the sum of the Hamming distances from the (M-1) distinct combinations of two vertices, where each pair of vertices is X_i and a vertex in F that is not X_i, is a value Q_i defined as

$$Q_i = \sum_{j=1}^{n} x_{ij}\ (M-C_j) + (1-x_{ij})\ C_j \quad .$$

4.2 Computational Methods

A threshold function, composed of $n \geq 1$ variables and $M \geq 2$ vertices, will tend to have a lower mutual distance total than a non-threshold function composed of the same number of variables and vertices. For notational convenience,

$q(M,n)$ is used to represent the minimum

mutual distance total of all Boolean functions, composed of n variables and M vertices, that are neither 1-monotonic nor dual-comparable. $Q(M,n)$ is the maximum mutual distance total of all Boolean functions composed of n variables and M vertices. $Q_t(M,n)$ and $q_t(M,n)$ are, respectively, the maximum and minimum mutual distance totals of all Boolean functions, composed of n variables and M vertices, that are threshold. $Q_d(M,n)$ and $q_d(M,n)$ are respectively, the maximum and minimum mutual distance totals of all Boolean functions, composed of M vertices and n variables, that are not 1-monotonic but are dual-comparable. $Q_1(M,n)$ and $q_1(M,n)$ are, respectively, the maximum and minimum mutual distance totals of all Boolean functions, composed of n variables and M vertices, that are 1-monotonic but are neither 2-monotonic nor dual-comparable. Q_{d1} and q_{d1} are, respectively, the maximum and minimum mutual distance totals of all Boolean functions, composed of n variables and M vertices, that are 1-monotonic and dual-comparable, but are not 2-monotonic. $Q_2(M,n)$ and $q_2(M,n)$ are, respectively, the maximum and minimum mutual distance totals of all Boolean functions, composed of n variables and M vertices, that are 2-monotonic but are neither 3-monotonic nor dual-comparable. $Q_{d2}(M,n)$ and $q_{d2}(M,n)$ are, respectively, the maximum and minimum mutual distance totals of all Boolean functions, composed of n variables and M vertices, that are 2-monotonic

and dual-comparable but are not 3-monotonic.

The maximum mutual distance total $Q(M,n)$ may be calculated using the formula

$$Q(M,n) = I \cdot J \cdot n,$$

where I is the largest integer $\leq M/2$ and $J=M-I$. This is true since $K(M-K)$, $0 \leq K \leq M$, is largest only when

$$\begin{aligned} K = I \quad &\text{and} \quad (M-K) = J \\ \text{or} \quad K = J \quad &\qquad\quad (M-K) = I. \end{aligned}$$

One of the methods that was used to calculate some of these maximum and minimum mutual distance totals, was to generate all of the Boolean functions of three and four variables. All of the Boolean functions of five variables were not generated because there are 2^{32} Boolean functions of five variables; in fact, this method of generating all of the Boolean functions of n variables is practical only for small n.

Winder (1965b) gives an algorithm to generate all 2-monotonic, canonical, self-dual functions of n variables. This algorithm was modified to generate all of the 1-monotonic Boolean functions of four and five variables. To generate all of the 1-monotonic Boolean functions of n variables, an unspecified truth table, which is a vector of 2^n locations, is initially required. Each location I,

$1 \leq I \leq 2^n$, of this truth table corresponds to an n-tuple or a vertex $X = x_1 \cdots x_n$ which has a binary equivalent of (I-1); location I of the truth table has a value F(X). F(X) has a value 1 iff vertex $X \epsilon F$; F(X) has a value 0 iff vertex $X \epsilon \overline{F}$. If F(X) is unspecified, then vertex X has not, at this time, been assigned to either F or $\overline{F}$. A standard lattice L_1 defined on the unit n-cube is as follows: For any two vertices $X = x_1 \cdots x_n$ and $Y = y_1 \cdots y_n$, $X \leq Y$ means that wherever $x_i = 1$, $y_i = 1$ also. (See Figure 2 for n=3. The arrows point in the direction of increase.)

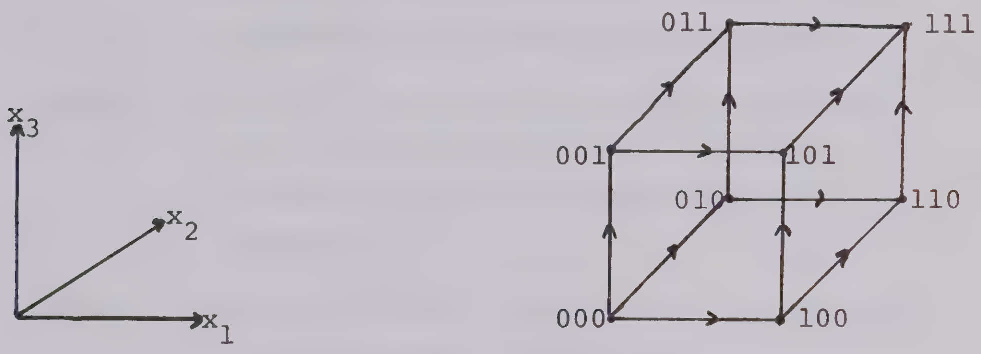

Figure 2. Standard lattice L_1 on the 3-cube

A set of vertices E, which is a subset of the vertices of a lattice, is called complete when a vertex $X \epsilon E$ and vertex $X \leq$ vertex Y imply that $Y \epsilon E$. The algorithm, to generate all of the 1-monotonic Boolean functions of n-variables, consists of seven steps:

Step 1: Find the highest unspecified X (highest in the ordering of L_1). If there are none,

go to step 4.

Step 2: Store a copy of the truth table in a pile with last-in-first-out store (for future assignment of this F(X) to 1).

Step 3: Assign $F(Y) = 0$ for all $Y \leq X$ according to the standard lattice L_1. Go back to step 1.

Step 4: Determine the properties, the mutual distance total, and the number of vertices of the 1-monotonic function which has been generated. Adjust, if necessary, the appropriate maximum and minimum mutual distance total arrays.

Step 5: Recover a previous partially specified truth table from the pile. If there are none, stop (the enumeration is complete).

Step 6: Find the highest unspecified X according to the standard lattice L_1.

Step 7: Assign $F(X) = 1$. Go back to step 1.

This algorithm, essentially, moves a "pointer" down the truth table; whenever an X is found for which F(X) is unspecified (Step 1), two possibilities must be explored: $F(X) = 0$ or 1. The first one considered is $F(X) = 0$ (Step 3); the alternative is considered later (Steps 6 and 7). Whenever F(X) is assigned to 0, certain other values of F,

or the truth table, must also be specified to maintain the completeness condition (Step 3).

Similarly, it is possible to modify the algorithm that is given by Winder (1965b) so as to develop an algorithm to generate all 2-monotonic, canonical Boolean functions of n variables. A new lattice L_2 must be defined on the n-cube. If $\dot{X}$, with components $\dot{x}_i$, i=1, ..., n, is an n-tupe of integers defined by

$$\dot{x}_i = \sum_{j=1}^{i} x_j,$$

then, for the new lattice L_2, vertex X$\leq$ vertex Y means that $\dot{X}_i \leq \dot{Y}_i$, i=1, ..., n.

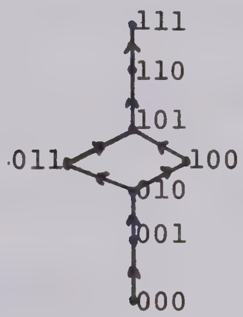

Figure 3. Lattice L_2 for n=3.

The algorithm, to generate all 2-monotonic, canonical Boolean functions of n variables, is:

Step 1: Find the highest unspecified X (highest in the ordering of L_2). If there are none, go to step 4.

Step 2: Store a copy of the truth table in a pile with last-in-first-out store (for future assignment of this F(X) to 1).

Step 3: Assign $F(Y) = 0$ for all $Y \leq X$ according to the lattice L_2. Go back to step 1.

Step 4: Determine the properties, the mutual distance total, and the number of vertices of the 2-monotonic canonical function which has been generated. Adjust, if necessary, the appropriate maximum and minimum mutual distance total arrays.

Step 5: Recover a previously partially specified truth table from the pile. If there are none, stop (the enumeration is complete).

Step 6: Find the highest unspecified X (according to the lattice L_2).

Step 7: Assign $F(X) = 1$. Go back to Step 1.

4.3 Computational Results

Computational results are shown in the following tables. A blank location on a table indicates that no Boolean function exists with the properties specified by corresponding column and row in the table.

Table 2 indicates that $A_t(M,3)$ < minimum of $(a_d(M,3), a(M,3))$ for $M=2, \ldots, 6$; hence, for any Boolean

TABLE 2

MUTUAL DISTANCE TOTALS FOR BOOLEAN FUNCTIONS OF THREE VARIABLES

Number of Vertices, M	$q_t(M,3)$	$Q_t(M,3)$	$q_d(M,3)$	$Q_d(M,3)$	$q(M,3)$	$Q(M,3)$
2	1	1	2	2	3	3
3	4	4	6	6	6	6
4	8	9	12	12	10	12
5	16	16	18	18	18	18
6	25	25	26	26	27	27
7	36	36				36
8	48	48				48

function $F(x_1,x_2,x_3)$, it is possible to determine whether or not F is threshold using the mutual distance total of F and M(F). If the mutual distance total Q of a Boolean function $F(x_1,x_2,x_3)$, where $M(F) \geq 2$, is greater than Q_d (M,3), then F is not dual-comparable; if $Q > Q_t$ (M,3), then F is non-threshold.

Table 3 indicates that there is no non-threshold 1-monotonic Boolean function $F(x_1, \ldots, x_4)$ where $6 \geq M(F) \geq 10$. It is possible to determine if a 1-monotonic Boolean function $F(x_1, \ldots, x_4)$ is threshold using M(F) and the

TABLE 3

MUTUAL DISTANCE TOTALS FOR BOOLEAN FUNCTIONS OF FOUR VARIABLES

Number of Vertices, M	q_t (M,4)	Q_t (M,4)	q_l (M,4)	Q_l (M,4)	q_d (M,4)	Q_d (M,4)	q (M,4)	Q (M,4)
2	1	1			2	3	4	4
3	4	4			6	8	8	8
4	8	9			10	16	13	16
5	16	16			18	24	20	24
6	25	26			26	35	28	36
7	36	38	40	40	40	48	40	48
8	48	52	54	54	56	64	53	64
9	68	70	72	72	72	80	72	80
10	89	90			90	99	92	100
11	112	112			114	120	116	120
12	136	137			138	144	141	144
13	164	164			166	168	168	168
14	193	193			194	195	196	196
15	224	224						224
16	256	256						256

mutual distance total of F. If a Boolean function $F(x_1, \ldots, x_n)$, composed of $M \geq 2$ vertices, has a mutual distance total $Q >$ maximum of $(Q_1(M,4), Q_t(M,4))$, then F is not 1-monotonic; if $Q > Q_t\ (M,4)$, then F is not threshold. In fact, it is possible to determine if a Boolean function $F(x_1, \ldots, x_4)$, composed of m vertices (where $M \neq 6$ and $M \neq 10$), is threshold or not using only M and the mutual distance total of F.

TABLE 4

MUTUAL DISTANCE TOTALS FOR BOOLEAN FUNCTIONS OF FIVE VARIABLES

Number of Vertices, M	q_t (M,5)	Q_t (M,5)	q_{d1} (M,5)	Q_{d1} (M,5)	q_1 (M,5)	Q_1 (M,5)	Q (M,5)
2	1	1					5
3	4	4					10
4	8	9					20
5	16	16					30
6	25	26					45
7	36	38	40	40			60
8	48	53	54	55			80
9	68	70	72	74			100
10	89	91	93	95			125
11	112	114	116	120	120	120	150

TABLE 4, continued

Number of Vertices, M	q_t (M,5)	Q_t (M,5)	q_{d1} (M,5)	Q_{d1} (M,5)	q_1 (M,5)	Q_1 (M,5)	Q (M,5)
12	136	141	142	145	146	146	180
13	164	170	172	174	174	176	210
14	193	203	204	205	204	209	245
15	224	238			238	244	280
16	256	276			274	283	320
17	304	318			318	324	360
18	353	363	364	365	364	369	405
19	404	410	412	414	414	416	450
20	456	461	462	465	466	466	500
21	512	514	516	520	520	520	550
22	569	571	573	575			605
23	628	630	632	634			660
24	688	693	694	695			720
25	756	758	760	760			780
26	825	826					845
27	896	896					910
28	968	969					980
29	1,044	1,044					1,050
30	1,121	1,121					1,125
31	1,200	1,200					1,200
32	1,280	1,280					1,280

Table 4 indicates that there is no non-threshold, dual-comparable, 1-monotonic Boolean function $F(x_1, \ldots, x_5)$ where $M(F) = 15, 16$, or 17; hence, a self-dual, 1-monotonic Boolean function $F(x_1, \ldots, x_5)$ is necessarily threshold. If a Boolean function $F(x_1, \ldots, x_5)$ composed of $M \geq 2$ vertices, has a mutual distance total $Q > A_t(M,5)$, then F is not threshold. If Q > maximum of $(Q_t(M,5), Q_{d1}(M,5))$, then F is not 1-monotonic and dual-comparable; if Q > maximum of $(Q_t(M,5), Q_{d1}(M,5), Q_1(M,5))$, then F is not 1-monotonic. For any 1-monotonic Boolean function $F(x_1, \ldots, x_5)$ composed of M vertices $(15>M>17)$, it is possible to determine if F is threshold or not using the mutual distance total of F and M. For any dual-comparable, 1-monotonic Boolean function $F(x_1,\ldots,x_n)$, it is possible to determine whether F is threshold or not using the mutual distance total of F and M(F).

TABLE 5

MUTUAL DISTANCE TOTALS FOR BOOLEAN FUNCTIONS OF SIX VARIABLES

Number of Vertices, M	q_t (M,6)	Q_t (M,6)	q_2 (M,6)	Q_2 (M,6)	Q (M,6)
2	1	1			6
3	4	4			12
4	8	9			24
5	25	26			36

TABLE 5, continued

Number of Vertices, M	q_t (M,6)	Q_t (M,6)	q_2 (M,6)	Q_2 (M,6)	Q (M,6)
6	36	38			54
7	36	38			72
8	48	53			96
9	68	70			120
10	89	91			150
11	112	114			180
12	136	141			210
13	164	170			252
14	193	203			294
15	224	238			336
16	256	278			384
17	304	320			432
18	353	366			486
19	404	414			540
20	456	467			600
21	512	522			660
22	569	582			726
23	628	644			792
24	688	711			864
25	756	780			936

TABLE 5, continued

Number of Vertices,M	q_t (M,6)	Q_t (M,6)	q_2 (M,6)	Q_2 (M,6)	Q (M,6)
26	825	854			1,014
27	896	930	932	932	1,092
28	968	1,010	1,013	1,013	1,176
29	1,044	1,094	1,096	1,096	1,260
30	1,121	1,180	1,173	1,183	1,350
31	1,200	1,270	1,260	1,272	1,440
32	1,280	1,364	1,349	1,366	1,536
33	1,392	1,462	1,452	1,464	1,632
34	1,505	1,564	1,557	1,567	1,734
35	1,620	1,670	1,672	1,672	1,836
36	1,736	1,778	1,781	1,781	1,944
37	1,856	1,890	1,892	1,892	2.052
38	1,977	2,006			2,166
39	2,100	2,124			2,280
40	2,224	2,247			2,400
41	2,356	2,372			2,520
42	2,489	2.502			2,640
43	2,624	2,634			2,772
44	2,760	2,771			2,904
45	2,900	2,910			3,036

TABLE 5, continued

Number of Vertices, M	q_t (M,6)	Q_t (M,6)	q_2 (M,6)	Q_2 (M,6)	Q (M,6)
46	3,041	3,054			3,174
47	3,184	3,200			3,312
48	3,328	3,350			3,456
49	3,488	3,502			3,600
50	3,649	3,659			3,750
51	3,812	3,818			3,900
52	3,976	3,981			4,056
53	4,144	4,146			4,212
54	4,313	4,315			4,374
55	4,484	4,486			4,536
56	4,456	4,461			4,704
57	4,436	4,438			4,872
58	5,017	5,018			5,046
59	5,200	5,200			5,220
60	5,384	5,385			5,400
61	5,572	5,572			5,580
62	5,761	5,761			5,766
63	5,952	5,952			5,952
64	6,144	6,144			6,144

Table 5 indicates that there is no non-threshold, 2-monotonic Boolean function $F(x_1, \ldots, x_6)$ where $26 \geq M(F) \geq 38$. If a Boolean function $F(x_1, \ldots, x_6)$, composed of $M \geq 2$ vertices, has a mutual distance total $Q >$ maximum of $Q_t(M,6)$, $Q_2(M,6)))$, then F is not 2-monotonic; if $Q > Q_t(M,6)$, then F is not threshold. For any 2-monotonic Boolean function $F(x_1, \ldots x_6)$, composed of M vertices $(29 \geq M \geq 35)$, it is possible to determine if F is threshold or not using the mutual distance total of F and M.

TABLE 6

MUTUAL DISTANCE TOTALS FOR BOOLEAN FUNCTIONS OF SEVEN VARIABLES

Number of Vertices, M	q_t (M,7)	Q_t (M,7)	q_{d2} (M,7)	Q_{d2} (M,7)	q_2 (M,7)	Q_2 (M,7)	Q (M,7)
2	1	1					7
3	4	4					14
4	8	9					28
5	16	16					42
6	25	26					63
7	36	38					84
8	48	53					112
9	68	70					140
10	89	91					175
11	112	114					210

TABLE 6, continued

Number of Vertices, M	q_t (M,7)	Q_t (M,7)	q_{d2} (M,7)	Q_{d2} (M,7)	q_2 (M,7)	Q_2 (M,7)	Q (M,7)
12	136	141					252
13	164	170					294
14	193	203					343
15	224	238					392
16	256	278					448
17	304	320					504
18	353	366					567
19	404	414					630
20	456	467					700
21	512	522					770
22	569	582					847
23	628	644					924
24	688	711					1,008
25	756	780					1,092
26	825	854					1,183
27	896	930	932	932			1,274
28	968	1,011	1,011	1,013			1,372
29	1,044	1,094	1,096	1,098			1,470
30	1,121	1,183	1,173	1,187			1,575
31	1,200	1,274	1,260	1,278			1,680
32	1,280	1,371	1,349	1,374			1,792

TABLE 6, continued

Number of Vertices, M	q_t (M,7)	Q_t (M,7)	q_{d2} (M,7)	Q_{d2} (M,7)	q_2 (M,7)	Q_2 (M,7)	Q (M,7)
33	1,392	1,470	1,452	1,474			1,904
34	1,505	1,574	1,557	1,579			2,023
35	1,620	1,680	1,666	1,688			2,142
36	1,736	1,792	1,777	1,799			2,268
37	1,856	1,906	1,892	1,914			2,394
38	1,977	2,024	2,011	2,036			2,527
39	2,100	2,146	2,134	2,158			2,660
40	2,224	2,272	2,261	2,282			2,800
41	2,356	2,402	2,390	2,410			2,940
42	2,489	2,536	2,525	2,544			3,087
43	2,624	2,674	2,662	2,682			3,234
44	2,760	2,816	2,801	2,824			3,388
45	2,900	2,962	2,944	2,972	2,962	2,962	3,542
46	3,041	3,112	3,089	3,120	3,115	3,115	3,703
47	3,184	3,266	3,238	3,274	3,270	3,270	3,864
48	3,328	3,424	3,389	3,434	3,427	3,429	4,032
49	3,488	3,586	3,550	3,596	3,586	3,592	4,200
50	3,649	3,752	3,713	3,760	3,751	3,758	4,375
51	3,812	3,922	3,880	3,932	3,916	3,926	4,550
52	3,976	4,096	4,049	4,108	4,087	4,098	4,732

TABLE 6, continued

Number of Vertices, M	q_t (M,7)	Q_t (M,7)	q_{d2} (M,7)	Q_{d2} (M,7)	q_2 (M,7)	Q_2 (M,7)	Q (M,7)
53	4,144	4,274	4,222	4,286	4,260	4,276	4,914
54	4,313	4,456	4,397	4,465	4,435	4,457	5,103
55	4,484	4,642	4,576	4,562	4,606	4,646	5,292
56	4,656	4,832	4,757	4,841	4,792	4,837	5,488
57	4,836	5,026	4,944	5,034	4,980	5,032	5,684
58	5,017	5,223	5,150	5,230	5,170	5,229	5,887
59	5,200	5,424	5,344	5,430	5,362	5,432	6,090
60	5,384	5,630	5,540	5,636	5,561	5,639	6,300
61	5,572	5,840	5,758	5,842	5,716	5,850	6,510
62	5,761	6,053	5,978	6,050	5,917	6,065	6,727
63	5,952	6,270			6,120	6,282	6,944
64	6,144	6,492			6,325	6,503	7,168
65	6,400	6,718			6,568	6,730	7,392
66	6,657	6,949	6,874	6,946	6,813	6,961	7,623
67	6,916	7,184	7,102	7,186	7,060	7,194	7,854
68	7,176	7,422	7,332	7,428	7,353	7,431	8,092
69	7,440	7,664	7,584	7,670	7,602	7,672	8,330
70	7,705	7,911	7,838	7,918	7,858	7,917	8,575
71	7,972	8,162	8,080	8,170	8,116	8,168	8,820
72	8,240	8,416	8,341	8,425	8,376	8,421	9,072

TABLE 6, continued

Number of Vertices, M	q_t (M,7)	Q_t (M,7)	q_{d2} (M,7)	Q_{d2} (M,7)	q_2 (M,7)	Q_2 (M,7)	Q (M,7)
73	8,516	8,674	8,608	8,684	8,638	8,678	9,324
74	8,793	8,936	8,877	8,945	8,915	8,937	9,583
75	9,072	9,202	9,150	9,214	9,188	9,204	9,842
76	9,352	9,472	9,425	9,484	9,463	9,474	10,108
77	9,636	9,746	9,704	9,756	9,470	9,750	10,374
78	9,921	10,024	9,985	10,032	10,023	10,030	10,647
79	10,208	10,306	10,270	10,316	10,306	10,312	10,920
80	10,496	10,592	10,557	10,602	10,595	10,597	11,200
81	10,800	10,882	10,854	10,890	10,886	10,886	11,480
82	11,105	11,176	11,153	11,184	11,179	11,179	11,767
83	11,412	11,474	11,456	11,484	11,474	11,474	12,054
84	11,720	11,776	11,761	11,784			12,348
85	12,032	12,082	12,070	12,090			12,642
86	12,345	12,392	12,381	12,400			12,943
87	12,660	12,706	12,694	12,714			13,244
88	12,976	13,024	13,013	13,034			13,552
89	13,300	13,346	13,334	13,358			13,860
90	13,625	13,672	13,659	13,684			14,175
91	13,952	14,002	13,988	14,010			14,490
92	14,280	14,336	14,321	14,343			14,812

TABLE 6, continued

Number of Vertices,M	q_t (M,7)	Q_t (M,7)	q_{d2} (M,7)	Q_{d2} (M,7)	q_2 (M,7)	Q_2 (M,7)	Q (M,7)
93	14,612	14,672	14,658	14,680			15,134
94	14,945	15,014	14,997	15,019			15,463
95	15,280	15,358	15,340	15,362			15,792
96	15,616	15,707	15,685	15,710			16,128
97	15,984	16,058	16,044	16,062			16,464
98	16,353	16,415	16,405	16,419			16,807
99	16,724	16,774	16,776	16,778			17,150
100	17,096	17,139	17,139	17,141			17,500
101	17,472	17,506	17,508	17,508			17,850
102	17,849	17,878					18,207
103	18,228	18,252					18,564
104	18,608	18,631					18,928
105	18,996	19,012					19,292
106	19,385	19,398					19,663
107	19,776	19,786					20,034
108	20,168	20,179					20,412
109	20,564	20,574					20,790
110	20,961	20,974					21,175
111	21,360	21,376					21,560
112	21,760	21,782					21,952
113	22,176	22,190					22,344

TABLE 6, continued

Number of Vertices, M	q_t (M,7)	Q_t (M,7)	q_{d2} (M,7)	Q_{d2} (M,7)	q_2 (M,7)	Q_2 (M,7)	Q (M,7)
114	22,593	22,603					22,743
115	23,012	23,018					23,142
116	23,432	23,437					23,548
117	23,856	23,858					23,954
118	24,281	24,283					24,367
119	24,708	24,710					24,780
120	25,136	25,141					25,200
121	25,572	25,574					25,620
122	26,009	26,010					26,047
123	26,448	26,448					26,474
124	26,888	26,889					26,908
125	27,332	27,332					27,342
126	27,777	27,777					27,783
127	28,224	28,224					28,224
128	28,672	28,672					28,672

Table 6 indicates that there is no non-threshold, dual-comparable, 2-monotonic Boolean function $F(x_1, \ldots, x_7)$ where $M(F) = 63$, 64 or 65; hence, a self-dual, 2-monotonic Boolean function $F(x_1 \ldots, x_7)$ is necessarily threshold. If a Boolean function $F(x_1 \ldots, x_7)$, composed of $M \geq 2$ vertices, has a mutual distance total $Q > A_t(M,7)$, then F is non-

threshold. If $Q >$ maximum of $Q_t(M,7)$, $Q_{d2}(M,7)$, then F is not 2-monotonic and dual-comparable; if $Q >$ maximum of $(Q_t(M,7)$, $Q_{d2}(M,7)$, $Q_2(M,7)$, then F is not 2-monotonic.

4.4 Final Remarks and Suggestions for Further Research

In this thesis, Hamming distance has been used to define several distance parameters for a Boolean function $F(x_1,\ldots,x_n)$ in a Boolean n-space; the distance parameters d, $\bar{d}$, $\bar{d}'$, D and Q can help to make the task of determining the properties of F much easier.

In addition to these results, several other interesting facts were found.

(1) It is known that all of the Boolean functions $F(x_1,\ldots,x_n)$ within a symmetric class share the same Boolean properties and are labelled by their Chow parameters. It has been shown that all Boolean functions within a symmetric class share the same distance vector. However, it is possible for two Boolean functions $F(x_1,\ldots,x_n)$ and $G(x_1,\ldots,x_n)$, where F is threshold and G is not threshold, to have the same distance vector. E.g., consider,

$$F(x_1,\ldots,x_4) = \bar{x}_1\bar{x}_2 + \bar{x}_1\bar{x}_3\bar{x}_4 + \bar{x}_2\bar{x}_3\bar{x}_4$$

and $$G(x_1,\ldots,x_4) = \bar{x}_1\bar{x}_2 + \bar{x}_1\bar{x}_3\bar{x}_4 + \bar{x}_1x_3x_4.$$

Hence, a distance vector, unlike a canonical form of the Chow parameters, cannot be used to label threshold functions.

(2) An unsuccessful attempt was made to derive a relatively fast method to calculate $Q_t(m,n)$ and

$q(m,n)$; however, this method did provide a good approximation of $Q_t(m,n)$ and $q(m,n)$. For example, to find the Boolean function from which $Q_t(m,n)$ could be calculated, a vertex was added to any Boolean function $F(x_1,\ldots,x_n)$ from which Q_t (m-1, n) could be calculated; the selection of this vertex involved testing the (2^n-m+1) vertices in $\overline{F}$. Hence, the order of calculation would be, from first to last, $Q_t(2,n)$, $Q_t(3,n),\ldots,Q_t(2^n,n)$. Similarly, an approximation of $q(m,n)$ can be calculated.

(3) It took a computer program, written in FORTRAN II and run on a PDP-9 computer, approximately eight hours to generate all of the 1-monotonic Boolean functions of five variables; also, it took another computer program, written in FORTRAN II and run on a PDP-9 computer, approximately twelve hours to generate all of the canonoical 2-monotonic functions of seven variables. The execution time for these programs increases exponentially as the number of variables increases.

The results of this investigation suggest several areas for future research. Some suggestions for future research include:

(1) The development of the theory and the algorithms necessary to provide a more efficient calculation of the distance parameters, so that these parameters can be calculated for Boolean functions with a larger number of variables.

(2) The development of further parameters which would be used with the distance parameters to give a more comprehensive description of the properties of any Boolean function $F(x_1,\ldots,x_n)$.

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